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The Imaginary Domains of Conics and their Interpretation in the Complex Plane

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THE IMAGINARY DOMAINS OF CONICS AND THEIR INTERPRETATION IN THE COMPLEX PLANE

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THE IMAGINARY DOMAINS OF CONICS AND THEIR INTERPRETATION IN THE COMPLEX PLANE.

INTRODUCTION.

It is the purpose of this thesis to make a study of the projective properties of conics extended to cases where the elements, as points and straight lines, entering the discussion become, partly or all, imaginary. For this purpose, a brief discussion, both geometric and analytic, of central projection and collineation and their properties precedes the study of such domains.

Although the principle results of such an extension, in one form or another, especially in connection with the theory of involution, have been known for a long time, it is not without value to derive the most important of them directly from the laws of central projection and the more general collineation and their analytic formulation. In this manner, a very clear insight into the relation between imaginary and real elements of conics is obtained. By imaginary domains of conics, we understand the totality of all imaginary elements, points or straight lines, and their relation with respect to conics.

In the third part, the variables of the equation of a conic are both assumed as the complex variables W and Z. In this case the discussion of the imaginary conic represented by such an equation is reduced to that of a special two valued function,

that is, to a problem of the theory of functions.

The case in which the domain of either x or y is that of the real x-axis or y-axis, respectively, is identical with that of the ordinary theory of conics.

A short historical account of the work done in developing the present idea of the imaginary elements concludes the work.

CHAPTER I.

SOME GENERAL PRINCIPLES OF CENTRAL PROJECTION AND COLLINEATION.

1. Geometrical Representation of Central Projection.

A central projection, or a perspective, is determined by a plane of projection and a point, called the center of projection.

Designate the plane of projection by π', which may be any plane in space, the center of projection by C, any point in space, and the plane of the figure to be projected by π, (Fig. 1). Let s be the line of intersection of π' with a plane through C parallel to π, and r, the line of intersection of π with a plane through C parallel to π'.

From Fig.(1), the projection of P in π is the point P', which is the point of intersection of the line joining C to P with the π' plane. Also the projection of l in π is the line l', the intersection of the plane through C and l with π'.

We can now state three fundamental laws:-

(a) To every point of π corresponds a point of π' and conversely, and both points lie on a ray through C.

(b) To every straight line of π corresponds a straight line of π' and conversely, and corresponding lines meet in a point of s.

(c) To the line at infinity of π corresponds the line q' of π', which is parallel to s. Conversely, to the line r' at infinity of π' corresponds a line r of π, which is parallel to s.

Since the planes through C are parallel to π and π', these

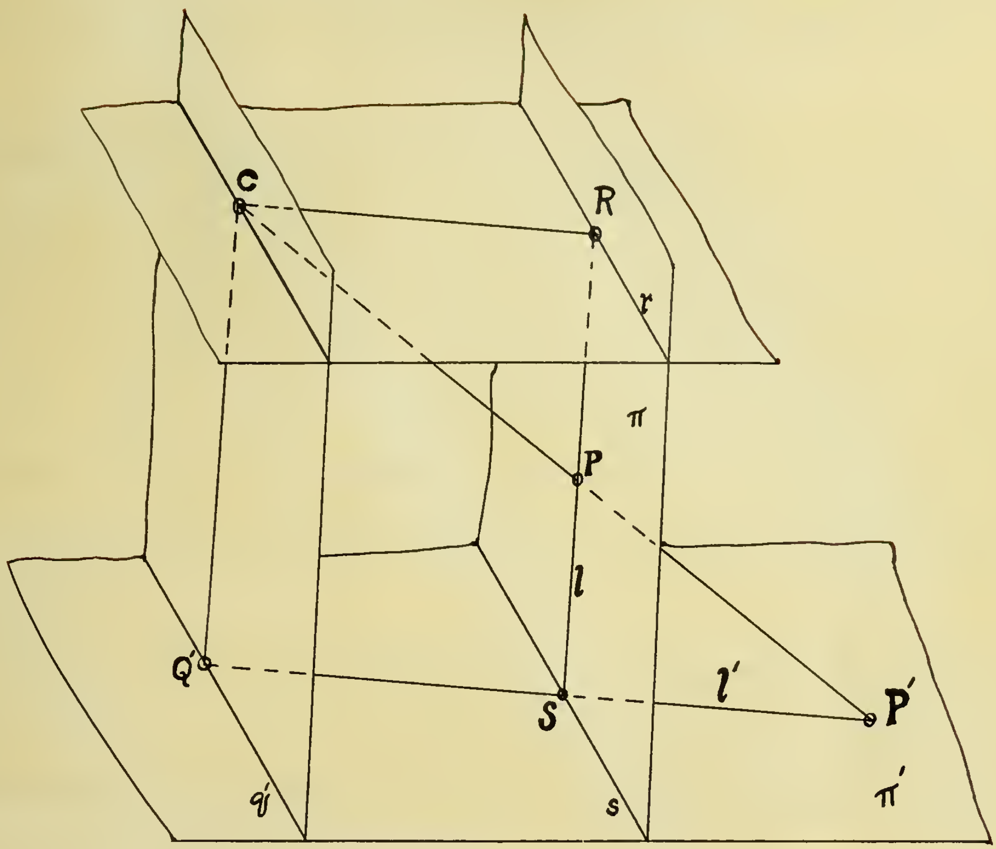

Fig. 1.

four planes form a space of a parallelopiped. Also the plane through l, C, and l' intersects these planes in a parallelogram, with sides CR equal and parallel to Q'S and CQ' equal and parallel to RS.

Therefore, keeping π' fixed, it is possible to turn the planes π and the two thru C parallel to π and π' down into π', without changing s and q' and the distances of C and r, C and q', S and r and S and q' in these planes. In this way, Fig. (1) is changed into Fig. (2). Then in Fig. (2), CR is equal and parallel to Q'S and the distance SP' is not changed, therefore the distances PR and PS are not changed by the rabattement. Also P and P' still lie on a ray through C.

In Fig. (2), l and l' are the two corresponding lines which with s and SC form a pencil of four rays through S. As this pencil is intersected by the rays CP and CQ, we have (CLP'P) = (CMQ'Q). Since Q is at infinity,

(CMQ'Q) = CQ'/MQ'= CO/NO, which is a constant and is entirely independent of the position of l, l', and CP.

Therefore (CLP'P) = constant, in which L is the intersection of any ray through C with s and P' and P the corresponding intersections with l' and l.

Therefore, since l may be any line and P any point in a plane, by means of this relation, the perspective of any figure in a plane may be constructed.

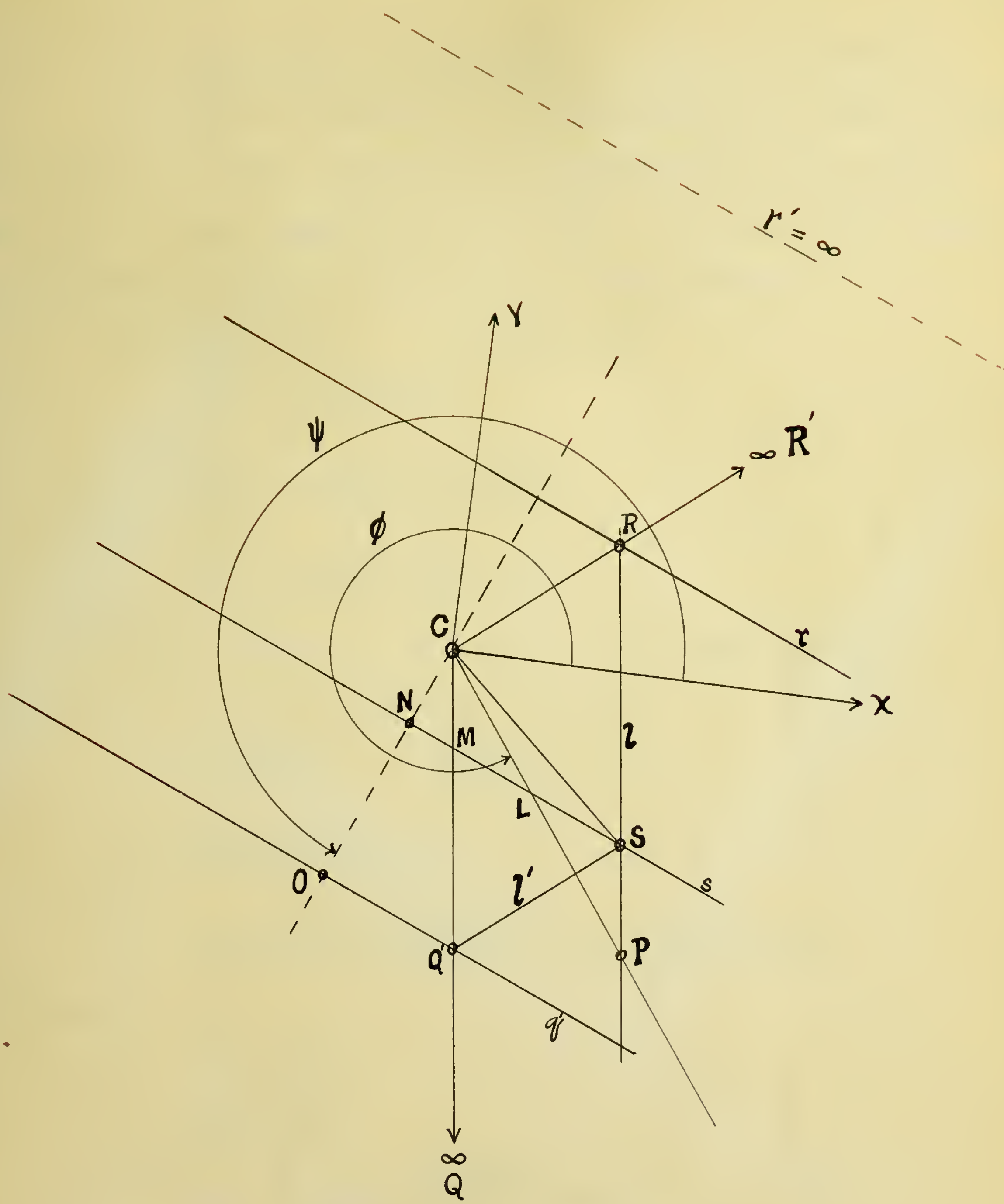

Fig. 2.

2. Analytical Representation of Central Projection.

In Fig. (2), assume any two perpendicular lines through C as coordinate axes and designate the angle made with the x-axis by CO by ψ and that made by CP by ϕ. Designate the coordinates of P and P' by (x,y) and (x',y') respectively.

We know (CLP'P) = k, a constant.

$$\therefore \quad CP' = \frac{k(CP \cdot LP')}{LP}$$

$$CP' = \frac{k \cdot CP(CP' - CL)}{CP - CL}$$

$$CP' = \frac{k \cdot CP \cdot CL}{(k-1)CP + CL}.$$

$CP = \sqrt{x^2 + y^2}$ and $CL = CN/\cos(\psi - \phi)$. (From Figure).

$$\cos(\psi - \phi) = \cos\psi\cos\phi + \sin\psi\sin\phi$$

$$= \frac{x\cos\psi}{\sqrt{x^2+y^2}} + \frac{y\sin\psi}{\sqrt{x^2+y^2}}.$$

$$CL = \frac{CN\sqrt{x^2+y^2}}{x\cos\psi + y\sin\psi}.$$

$$\therefore \quad CP' = \frac{k \cdot CN\sqrt{x^2+y^2}}{(k-1)\,x\cos\psi + (k-1)\,y\sin\psi + CN}.$$

From Figure.

$$x' = CP'\cos\phi,$$

$$y' = CP'\sin\phi.$$

$$\text{I}' \left[\begin{array}{l} x' = \dfrac{k \cdot CN \cdot x}{[(k-1)\cos\psi]x + [(k-1)\sin\psi]y + CN}, \\[2ex] y' = \dfrac{k \cdot CN \cdot y}{[(k-1)\cos\psi]x + [(k-1)\sin\psi]y + CN}. \end{array}\right.$$

By letting $k \cdot CN = a$, $(k-1)\cos\psi = d$, $(k-1)\sin\psi = e$, and $CN = f$, we get the equations of the perspective in the form:

$$\text{I} \left[\begin{array}{l} x' = \dfrac{ax}{dx + ey + f}, \\[2ex] y' = \dfrac{ay}{dx + ey + f}. \end{array}\right.$$

Solving these equations for x and y we get the following:-

$$\text{II}\quad \begin{cases} x = -\dfrac{fx'}{dx' + ey' - a}, \\[2ex] y = -\dfrac{fy'}{dx' + ey' - a}. \end{cases}$$

The most important cases of central projection may be classified according to the relative position of the center and axis and to the value of the constant k of the projection as follows:-

A. Involution

C is on the bisecting plane of π and π' and $k = -1$

The equations of transformation are:-

$$\text{III}\quad \begin{cases} x' = \dfrac{ax}{dx + ey - a}, \\[2ex] y' = \dfrac{ay}{dx + ey - a}. \end{cases}$$

B. Similitude.

The plane π' is parallel to π.

The equations of transformation are:-

$$\text{IV}\quad \begin{cases} x' = \dfrac{a}{f}x, \\[2ex] y' = \dfrac{a}{f}y. \end{cases}$$

If $k > 0$, π and π' are on the same side of C.

If $k < 0$, C is between the planes.

When $k = -1$, we have central symmetry.

C. Affinity.

C is at an infinite distance.

The equations of transformation are:-

$$V \left[\begin{array}{l} x' = x + ay, \\ y' = by. \end{array} \right.$$

3. General Collineation.

Collineation is that transformation which transforms each straight line and each point in a plane into a straight line and a point respectively.

The perspective transformation has this property, as was shown. Other transformations having this property are:-

Translation, represented by

$$\text{VI} \quad \begin{cases} x' = x + a, \\ y' = y + b; \end{cases}$$

Rotation, represented by

$$\text{VII} \quad \begin{cases} x' = x \cos\phi - y \sin\phi, \\ y' = x \sin\phi + y \cos\phi; \end{cases}$$

and Dilatation, represented by

$$\text{VIII} \quad \begin{cases} x' = \alpha x, \\ y' = \beta y. \end{cases}$$

Applying to the point (x,y) the Dilatation ($x' = \alpha x$, $y' = \beta y$) and then Affinity ($x'' = x' + \alpha_1 y'$, $y'' = \beta_1 y'$), the result is,

$$x'' = \alpha x + \alpha_1 \beta y,$$
$$y'' = \beta \beta_1 y.$$

Applying to this result the Rotation ($x''' = x'' \cos\phi - y'' \sin\phi$, $y''' = x'' \sin\phi + y'' \cos\phi$), the result is

$$x''' = \alpha \cos\phi \cdot x + (\alpha_1 \beta \cos\phi - \beta\beta_1 \sin\phi)\, y,$$
$$y''' = \alpha \sin\phi \cdot x + (\alpha_1 \beta \sin\phi + \beta\beta_1 \cos\phi)\, y.$$

Finally, applying the translation

($x^{IV} = x''' + \alpha_2$, $y^{IV} = y''' + \beta_2$), and letting

$x' = x^{IV}$ and $y' = y^{IV}$, the result is:-

[illegible]

[illegible]

[illegible]

$$\text{IX}' \quad \begin{cases} x' = \alpha\cos\phi\cdot x + (\alpha_1\beta\cos\phi - \beta\beta_1\sin\phi)\,y + \alpha_2, \\ y' = \alpha\sin\phi\cdot x + (\alpha_1\beta\sin\phi + \beta\beta_1\cos\phi)\,y + \beta_2, \end{cases}$$

which represents a combination of a dilatation, an affinity, a rotation and a translation.

IX' may be written in the form:-

$$\text{IX} \quad \begin{cases} x' = ax + by + c, \\ y' = dx + ey + f, \end{cases}$$

which is called a linear transformation. Since all of the constituent transformations are projective, a linear transformation is also projective, i. e. it transforms each straight line and each point in a plane into a straight line and a point respectively.

Now by combining the Linear Transformation ($x' = ax + by + c$, $y' = dx + ey + f$) with the Perspective Transformation ($x'' = \dfrac{a'x'}{d_1x' + e_1y' + f_1}$, $y'' = \dfrac{a'y'}{d_1x' + e_1y' + f_1}$), after dropping one prime, the result is:-

$$\text{X}' \quad \begin{cases} x' = \dfrac{a'ax + a'by + a'c}{(ad_1 + de_1)x + (bd_1 + ee_1)y + d_1c + e_1f + f_1} \\ y' = \dfrac{a'dx + a'ey + a'f}{(ad_1 + de_1)x + (bd_1 + ee_1)y + d_1c + e_1f + f_1}, \end{cases}$$

which can be put in the form:-

$$\text{X} \quad \begin{cases} x' = \dfrac{a_1x + b_1y + c_1}{a_3x + b_3y + c_3} \\ y' = \dfrac{a_2x + b_2y + c_2}{a_3x + b_3y + c_3}, \end{cases}$$

which is the most general collineation in a plane.

CHAPTER II.

THE EXTENSION OF THE PROJECTIVE PROPERTIES OF CONICS TO IMAGINARY DOMAINS.

1. Determination of the general collineation which transforms two conjugate imaginary points, J_1 and J_2, into the two circular points at infinity, I_1 and I_2.*

By the last theorem we proved that a collineation is determined by two given quadrangles. Therefore we can find a general collineation which will transform two given points into two other given points. Also as a special case, both sets of these points may be imaginary. To determine two conjugate imaginary points we may take the imaginary intersections of a conic with a straight line.

As a special problem then, we will find the general collineation which transforms the two conjugate imaginary points, determined by the imaginary intersections of the conic

$$K \equiv ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

with the x-axis $\equiv y = 0$, into the twocircular points $I_1(x_1' = \infty, y_1' = \infty, \frac{y_1'}{x_1'} = +i)$ and $I_2(x_2' = \infty, y_2' = \infty, \frac{y_2'}{x_2'} = -i)$.

Solving the equations simultaneously,

$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$, and $y = 0$, we get

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*The circular points at infinity were discovered by J. V. Poncelet in 1822. "Traité des propriétés projectives des figures". First Ed. Pages 48 and 49. Second Ed. Vol. I. Pages 47 and 48.

4. An Important Theorem concerning Collineation.

Any quadrangle may be transformed into any other quadrangle by a collineation.

Given any two quadrangles,

$$\left[A_1\,(x_1,y_1),\ A_2(x_2,y_2),\ A_3(x_3,y_3),\ A_4(x_4,y_4)\right] \text{ and}$$

$$\left[A'_1(x'_1\ y'_1),\ A'_2\ (x'_2,y'_2),\ A'_3(x'_3,y'_3),\ A'_4(x'_4,y'_4)\right].$$

To prove that (A_1, A_2, A_3, A_4) may be transformed into (A'_1, A'_2, A'_3, A'_4,) by a collineation.

Proof:- The general equation of collineation depends upon the eight parameters $\frac{a_1}{c_3}, \frac{b_1}{c_3}, \frac{c_1}{c_3}, \frac{a_2}{c_3}, \frac{b_2}{c_3}, \frac{c_2}{c_3}, \frac{a_3}{c_3}$, and $\frac{b_3}{c_3}$.

By substituting the coordinates of the eight points of the given quadrangles in the general equations of collineation, we get eight equations of condition by means of which the eight parameters are determined.

Therefore the collineation which transforms any given quadrangle into another given quadrangle can be uniquely determined. Furthermore, any quadrangle may be transformed into any other quadrangle by a collineation, for in the proof the given quadrangles were chosen arbitrarily.

$$x = \frac{-d \pm \sqrt{d^2 - fa}}{a},$$

$$y = 0.$$

The condition that the roots are imaginary is $af > d^2$. Therefore the imaginary roots and therefore the imaginary intersections of K with the x - axis are:-

$$J_1 \begin{cases} x_1 = \dfrac{-d + i\sqrt{af - d^2}}{a} & \text{and} \\ y_1 = 0 \end{cases}$$

$$J_2 \begin{cases} x_2 = \dfrac{-d - i\sqrt{af - d^2}}{a}, \\ y_2 = 0 \end{cases}$$

as shown in Fig. (3).

Therefore the problem is to find the collineation which transforms J_1 into I_1 and J_2 into I_2.

Given the general equations of Collineation

$$\begin{cases} x' = \dfrac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + c_3}, \\ y' = \dfrac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3}. \end{cases}$$

Since J_1 and J_2 are on the line $y = 0$, to transform these points to I_1 and I_2, this line must be transformed to the line at infinity.

From the general equations of transformation, in order to make $x' = \infty$ and $y' = \infty$, the denominators must equal zero. Therefore $a_3 x + b_3 y + c_3 = 0$, the equation of a straight line. But in this case, the equation of the straight line which is to

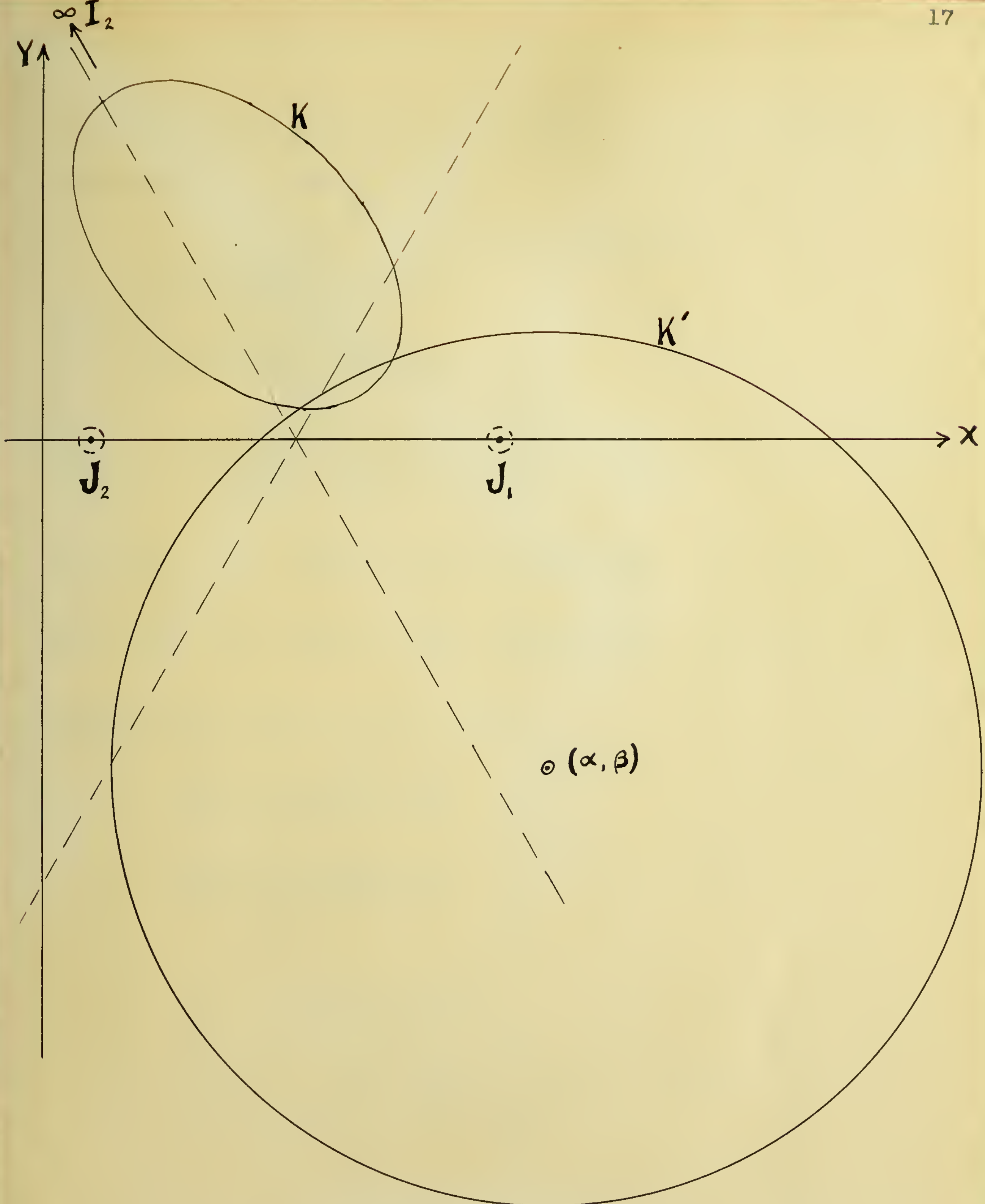

Fig. 3.

be transformed to the line at infinity is $y = 0$. Therefore the coefficients a_3 and c_3 vanish.

So the general equations reduce to:-

$$(1.)\quad \left[\begin{aligned} x' &= \frac{a_1x + b_1y + c_1}{b_3y} \\ y' &= \frac{a_2x + b_2y + c_2}{b_3y}. \end{aligned}\right.$$

Then $$\frac{y'}{x'} = \frac{a_2x + b_2y + c_2}{a_1x + b_1y + c_1}. \qquad (2.)$$

But $y'_1/x'_1 = +i$ and $x_1 = \dfrac{-d + i\sqrt{af - d^2}}{a}$, $y_1 = 0$,

and $y'_2/x'_2 = -i$ and $x_2 = \dfrac{-d - i\sqrt{af - d^2}}{a}$, $y_2 = 0$.

Substituting these values in (2.),

$$\left[\begin{aligned} &\frac{\dfrac{-a_2d + ia_2\sqrt{af - d^2}}{a} + c_2}{\dfrac{-a_1d + ia_1\sqrt{af - d^2}}{a} + c_1} = +i, \\[2ex] &\frac{\dfrac{-a_2d - ia_2\sqrt{af - d^2}}{a} + c_2}{\dfrac{-a_1d - ia_1\sqrt{af - d^2}}{a} + c_1} = -i. \end{aligned}\right.$$

$$\left[\begin{aligned} &\frac{-a_2d + ia_2\sqrt{af - d^2} + ac_2}{-a_1d + ia_1\sqrt{af - d^2} + ac_1} = +i \\[2ex] &\frac{-a_2d - ia_2\sqrt{af - d^2} + ac_2}{-a_1d - ia_1\sqrt{af - d^2} + ac_1} = -i \end{aligned}\right.$$

$$\begin{cases} -a_2d + ia_2\sqrt{af - d^2} + ac_2 = -ia_1d - a_1\sqrt{af - d^2} + iac_1 \\ -a_2d - ia_2\sqrt{af - d^2} + ac_2 = ia_1d - a_1\sqrt{af - d^2} - iac_1. \end{cases}$$

Adding, $\quad -2a_2d + 2ac_2 = -2a_1\sqrt{af - d^2}$,

Subtracting, $\quad 2ia_2\sqrt{af - d^2} = -2ia_1d + 2iac_1$,

Which reduce to $\quad a_2d - ac_2 = a_1\sqrt{af - d^2}, \qquad (3.)$

$$a_2\sqrt{af - d^2} = ac_1 - da_1. \qquad (4.)$$

From (4.) $\quad a_2 = \dfrac{ac_1 - da_1}{\sqrt{af - d^2}}.$

From (3.) $\quad c_2 = \dfrac{a_2d - a_1\sqrt{af - d^2}}{a}$

$$c_2 = \frac{\dfrac{dac_1 - d^2a_1}{\sqrt{af - d^2}} - a_1\sqrt{af - d^2}}{a}$$

$$c_2 = \frac{adc_1 - d^2a_1 - afa_1 + d^2a_1}{a\sqrt{af - d^2}}$$

Therefore $\quad c_2 = \dfrac{dc_1 - fa_1}{\sqrt{af - d^2}}.$

Substituting these values of a_2 and c_2 in (1.),

$$\begin{cases} x' = \dfrac{a_1x + b_1y + c_1}{b_3y}, \\ y' = \dfrac{\dfrac{ac_1 - da_1}{\sqrt{af - d^2}}x + b_2y + \dfrac{dc_1 - fa_1}{\sqrt{af - d^2}}}{b_3y}. \end{cases}$$

Dividing by b_3 and letting $a_1/b_3 = l$, $b_1/b_3 = k$, $c_1/b_3 = m$, and $b_2/b_3 = n$,

$$A \begin{cases} x' = \dfrac{lx + ky + m}{y}, \\ y' = \dfrac{\dfrac{am - dl}{\sqrt{af - d^2}}x + ny + \dfrac{dm - fl}{\sqrt{af - d^2}}}{y}. \end{cases}$$

This collineation contains four arbitrary independent parameters, k, l, m, and n. Therefore, there are ∞^4 transformations of the prescribed kind which transform J_1 and J_2 into I_1 and I_2, respectively.

Solving the above equations for x and y, we get:-

$$yx' = lx + ky + m,$$

$$yy' = \frac{am - dl}{\sqrt{af - d^2}}\, x + ny + \frac{dm - fl}{\sqrt{af - d^2}}.$$

$$(x' - k)\, y - lx - m = 0,$$

$$(y' - n)\sqrt{af - d^2}\, y - (am - dl)\, x - (dm - fl) = 0.$$

$$(x' - k)(am - dl)y - (am - dl)lx - (am - dl)m = 0,$$

$$(y' - n)l\sqrt{af - d^2}\, y - (am - dl)lx - (dm - fl)l = 0.$$

Subtracting,

$$\left[(x' - k)(am - dl) - (y' - n)l\sqrt{af - d^2}\right] y + (dm - fl)l - (am - dl)m = 0.$$

$$\left[\begin{aligned} y &= \frac{am^2 - 2dlm + fl^2}{(x' - k)(am - dl) - (y' - n)\, l\sqrt{af - d^2}} \\ x &= \frac{(x' - k)(am^2 - 2dlm + fl^2)}{(x' - k)(am - dl)l - (y' - n)\, l^2\sqrt{af - d^2}} - \frac{m}{l}. \end{aligned}\right.$$

$$B\left[\begin{aligned} x &= \frac{(am^2 - 2dlm + fl^2)x' - (am^2 - 2dlm + fl^2)k}{l(am - dl)x' - l^2\sqrt{af - d^2}\, y' - amlk + dl^2k + l^2n\sqrt{af - d^2}} - \frac{m}{l} \\ y &= \frac{am^2 - 2dlm + fl^2}{(am - dl)x' - l\sqrt{af - d^2}\, y' - amk + dlk + ln\sqrt{af - d^2}}, \end{aligned}\right.$$

which also contains the four arbitrary parameters k, l, m, and n and therefore there are ∞^4 converse transformations. For the sake of simplicity assume the case where k = 0 and n = 0. Then A becomes I and B reduces to II.

$$\text{I}\quad \begin{cases} x' = \dfrac{lx + m}{y} \\[2ex] y' = \dfrac{(am - dl)x + (dm - fl)}{\sqrt{af - d^2}\, y} \end{cases}$$

$$\text{II}\quad \begin{cases} x = \dfrac{(am^2 - 2dlm + fl^2)x'}{l(am - dl)x' - l^2\sqrt{af - d^2}\, y'} - \dfrac{m}{l} \\[3ex] y = \dfrac{am^2 - 2dlm + fl^2}{(am - dl)x' - l\sqrt{af - d^2}\, y'} \end{cases}$$

2. To investigate the equation of the conic K after it has been transformed by means of II.

Substitute the values given in II in $K \equiv ax^2 + 2\,bxy + cx^2 + 2dx + 2ey + f = 0$, and dropping primes, we get $K' \equiv$

$$\frac{a(am^2 - 2dlm + fl^2)^2\, x^2}{l^2(am - dl)^2x^2 - 2\, l^3(am - dl)\sqrt{af - d^2}\, xy + l^4(af - d^2)\, y^2}$$

$$- \frac{2\, am\,(am^2 - 2dlm + fl^2)\, x}{l^2(am - dl)\, x - l^3\sqrt{af - d^2}\, y} + \frac{am^2}{l^2}$$

$$+ \frac{2\, b(am^2 - 2dlm + fl^2)^2\, x}{l(am - dl)^2\, x^2 - 2\, l^2(am - dl)\sqrt{af - d^2}\, xy + l^3\,(af - d^2)\, y^2}$$

$$- \frac{2\, bm\,(am^2 - 2dlm + fl^2)}{l\,(am - dl)x - l^2\sqrt{af - d^2}\, y}$$

$$+\ \frac{c(am^2 - 2dlm + fl^2)^2}{(am - dl)^2 x^2 - 2\,l(am - dl)\sqrt{af - d^2}\,xy + l^2(af - d^2)\,y^2}$$

$$+\ \frac{2\,d(am^2 - 2dlm + fl^2)\,x}{l(am - dl)\,x - l^2\sqrt{af - d^2}\,y} - \frac{2dm}{l} + \frac{2e(am^2 - 2dlm + fl^2)}{(am - dl)x - l\sqrt{af - d^2}\,y}$$

$$+ f = 0.$$

Clearing of Fractions and combining terms,

$$\Big[al(am^2 - 2dlm + fl^2)^2 - 2\,aml(am^2 - 2\,dlm + fl^2)(am - dl)$$
$$+ am^2l(am - dl)^2 + 2\,dl^2(am^2 - 2dlm + fl^2)(am - dl) - 2\,dml^2(am - dl)^2$$
$$+ fl^3(am - dl)^2\Big]x^2 + \Big[2aml^2(am^2 - 2dlm + fl^2)\sqrt{af - d^2} - 2\,am^2l^2$$
$$(am - dl)\sqrt{af - d^2} - 2\,dl^3(am^2 - 2dlm + fl^2)\sqrt{af - d^2}$$
$$+ 4\,dml^3(am - dl)\sqrt{af - d^2} - 2fl^4(am - dl)\sqrt{af - d^2}\Big]xy$$
$$+ \Big[l^3am^2(af - d^2) - 2\,dml^4(af - d^2) + fl^5(af - d^2)\Big]y^2$$
$$+ \Big[2\,bl^2(am^2 - 2\,dlm + fl^2)^2 - 2\,bml^2(am - dl)(am^2 - 2\,dlm + fl^2)$$
$$+ 2\,el^3(am^2 - 2\,dlm + fl^2)(am - dl)\Big]x + \Big[2\,bml^3(am^2 - 2dlm + fl^2)$$
$$\sqrt{af - d^2} - 2\,el^4(am^2 - 2\,dlm + fl^2)\sqrt{af - d^2}\Big]y + cl^3$$
$$(am^2 - 2dlm + fl^2)^2 = 0.$$

Expanding and combining terms,

$$\Big[a^2m^2l^3f - am^2d^2l^3 - 2\,adml^4f + 2\,d^3ml^4 + af^2l^5 - d^2fl^5\Big]x^2$$
$$+ \Big[a^2m^2l^3f - am^2d^2l^3 - 2\,adml^4f + 2\,d^3ml^4 + af^2l^5 - d^2f\,l^5\Big]y^2$$
$$+ \Big[2\,ab\,l^4m^2f - 2\,abdl^3m^3 + 4\,bd^2l^4m^2 - 6\,bdl^5mf + 2\,bf^2l^6$$
$$+ 2\,a^2el^3m^3 - 6\,adel^4m^2 + 2\,aefl^5m + 4\,d^2el^5m - 2\,defl^6\Big]x + \Big[2abl^3m^3$$
$$- 4\,bdl^4m^2 + 2bfl^5m - 2\,ael^4m^2 + 4\,del^5m - 2\,efl^6\Big]\sqrt{af - d^2}\,y$$
$$+ a^2cl^3m^4 - 4\,acdl^4m^3 + 2\,acfl^5m^2 + 4\,cd^2l^5m^2$$
$$- 4\,cdfl^6m + cf^2l^7 = 0,$$

which is the equation of a circle, for the coefficients of x^2 and y^2 are equal and the coefficient of the xy term vanished.

But since there are ∞^4 transformations which transform a given conic into a circle, the conic may be transformed into ∞^4 circles.

The center of the circle is:

$$\alpha = -\tfrac{1}{2}D = -\frac{1}{2}\,\frac{2(am^2l^3 - 2\,dml^4 + fl^5)(lbf - bdm + mae - del)}{(am^2l^3 - 2\,dml^4 + fl^5)(af - d^2)}$$

$$= \frac{lde - lbf + mbd - mae}{af - d^2}.$$

$$\beta = -\tfrac{1}{2}E = -\frac{1}{2}\,\frac{2(am^2l^3 - 2\,dml^4 + fl^5)(mb - le)\sqrt{af - d^2}}{(am^2l^3 - 2\,dml^4 + fl^5)\,(af - d^2)} = \frac{le - mb}{\sqrt{af - d^2}}.$$

3. A Discussion of the System of Conics which is doubly tangent at J_1 and J_2 and the corresponding system derived by transformation II.

We must first extablish the condition for a system of conics to be tangent to each other at two given points.

In Fig. (4.), given the system of conics $gh + \lambda l^2 = 0$, which pass through T_1 and T_2. T_1 and T_2 are determined by the intersections of $h = 0$ and $g = 0$ respectively with $l^2 = 0$.

Taking the derivative of $gh + \lambda l^2 = 0$,

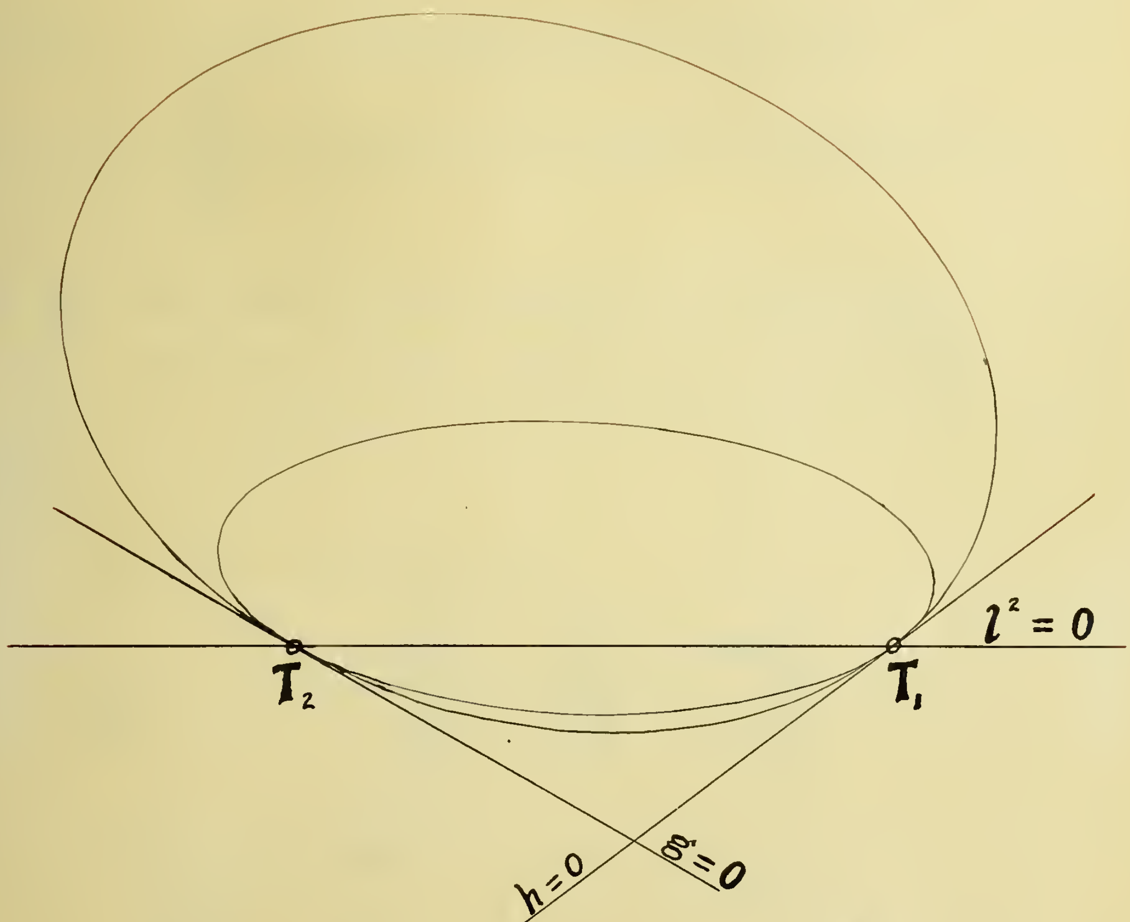

Fig. 4.

$$\frac{dy}{dx} = - \frac{g\frac{\delta h}{\delta x} + h\frac{\delta g}{\delta x} + 2\lambda l\frac{\delta l}{\delta x}}{g\frac{\delta h}{\delta y} + h\frac{\delta g}{\delta y} + 2\lambda l\frac{\delta l}{\delta y}} .$$

But when $l = 0$ and $h = 0$, that is at the point T_1,

$$\frac{dy}{dx} = - \frac{\frac{\delta h}{\delta x}}{\frac{\delta h}{\delta y}}$$, which is the slope of the line $h = 0$.

Since $\frac{dy}{dx}$ is the slope at any point in the above system of conics, and at T_1 the slope of every conic is the same and equal to the slope of $h = 0$, therefore the conics are tangent to each other and to $h = 0$ at T_1.

Again, take the derivative

$$\frac{dy}{dx} = - \frac{g\frac{\delta h}{\delta x} + h\frac{\delta g}{\delta x} + 2\lambda l\frac{\delta l}{\delta x}}{g\frac{\delta h}{\delta y} + h\frac{\delta g}{\delta y} + 2\lambda l\frac{\delta l}{\delta y}} .$$

When $l = 0$ and $g = 0$, that is at the point T_2,

$$\frac{dy}{dx} = - \frac{\frac{\delta g}{\delta x}}{\frac{\delta g}{\delta y}}$$, which is the slope of the line $g = 0$.

Therefore, since the slope of every conic of the system at T_2 is the same and equal to the slope of $g = 0$, all the conics of the system are tangent to each other and to $g = 0$ at T_2.

Therefore the system of conics is doubly tangent at T_1 and T_2.

In like manner the system of conics which is doubly tangent at J_1 and J_2 may be represented by the equation $K + \lambda y^2 = 0$, which may be shown as follows:-

Taking the derivative of $K + \lambda y^2 = 0$,

$$\frac{dy}{dx} = -\frac{\frac{\partial K}{\partial x}}{\frac{\partial K}{\partial y} + 2\lambda y}.$$

But when $y = 0$, that is at the points J_1 and J_2,

$$\frac{dy}{dx} = -\frac{\frac{\partial K}{\partial x}}{\frac{\partial K}{\partial y}}\text{, which is the slope of the conic } K.$$

Therefore the system of conics represented by $K + \lambda y^2 = 0$ has the same slope at J_1 and J_2 as the conic K has at those points, and therefore such a system is doubly tangent at those points.

To find the lines through J_1 and J_2 which are tangent to the system of conics $f(x,y) \equiv K + \lambda y^2 = 0$, or

$$f(x,y) \equiv ax^2 + 2\,bxy + cy^2 + 2\,dx + 2\,ey + f + \lambda y^2 = 0,$$

$$ax^2 + 2\,bxy + (c + \lambda)y^2 + 2\,dx + 2\,ey + f = 0.$$

To find the tangent through J_1 (x_1, y_1). The general equation of the tangent is $(x - x_1)\frac{\partial f}{\partial x} + (y - y_1)\frac{\partial f}{\partial y} = 0$.

$$\frac{\partial f}{\partial x} = 2\,ax_1 + 2\,by_1 + 2\,d.$$

$$\frac{\partial f}{\partial y} = 2\,bx_1 + 2(c + \lambda)y_1 + 2\,e.$$

$$x_1 = \frac{-d + i\sqrt{af - d^2}}{a} \text{ and } y_1 = 0.$$

$$\therefore \quad \frac{\partial f}{\partial x} = 2\,a\,\frac{(-d + i\sqrt{af - d^2})}{a} + 2\,d = 2\,i\sqrt{af - d^2}.$$

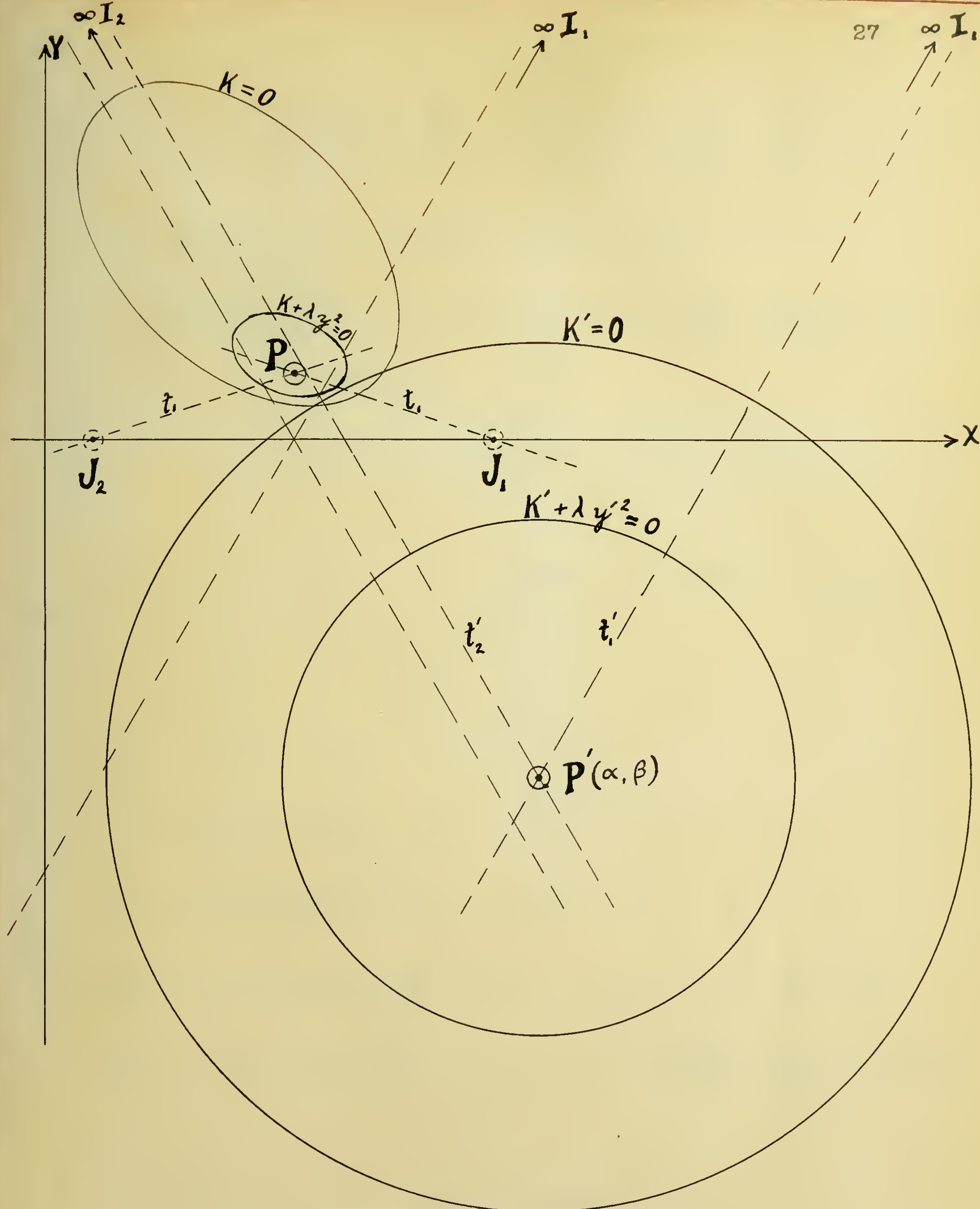

Fig. 5.

$$\frac{\delta f}{\delta y} = 2\ b\ \left(\frac{-d + i\sqrt{af - d^2}}{a}\right) + 2\ e = -\ \frac{2\ bd}{a} + \frac{2\ b}{a}\ i\ \sqrt{af - d^2}$$
$$+\ 2\ e.$$

$\therefore$ the equation of the tangent through J_1 is

$$\left(x - \frac{-d + i\sqrt{af - d^2}}{a} \right) 2\ i\ \sqrt{af - d^2} + y\left(-\ \frac{2\ bd}{a} + \frac{2\ b}{a}\ i\ \sqrt{af - d^2}\right.$$
$$\left. +\ 2\ e\right) = 0\ ,$$ which reduces to

$$i\sqrt{af - d^2}\ X + \frac{d}{a}\ i\ \sqrt{af - d^2} + \frac{af - d2}{a} - \frac{bd}{a}\ y + \frac{b}{a}\ i\ \sqrt{af - d^2}\ y$$
$$+\ ey = 0$$

To find the equation of the tangent through J2 (x_2, y_2).
The general equation of the tangent is

$$(x - x_2)\ \frac{\delta f}{\delta x} + (y - y_2)\frac{\delta f}{\delta y} = 0.$$

$$\frac{\delta f}{\delta x} = 2\ ax_2 + 2\ by_2 + 2\ d.$$

$$\frac{\delta f}{\delta y} = 2\ bx_2 + 2\ (c + \lambda)y_2 + 2\ e.$$

$$x_2 = \frac{-\ d - i\sqrt{af - d^2}}{a} \quad \text{and } y_2 = 0.$$

$$\therefore \frac{\delta f}{\delta x} = 2\ a\ \left(\frac{-d - i\sqrt{af - d^2}}{a}\right) + 2\ d = -\ 2\ i\ \sqrt{af - d^2}.$$

$$\frac{\delta f}{\delta y} = 2\ b\ \left(\frac{-d - i\sqrt{af - d^2}}{a}\right) + 2\ e = -\frac{2\ bd}{a} - \frac{2\ b}{a}\ i\sqrt{af - d^2}$$

$$+\ 2\ e.$$

$\therefore$ The equation of the tengent through J_2 is

$$\left(x - \frac{-\ d - i\sqrt{af - d^2}}{a}\right)\left(-2\ i\ \sqrt{af - d^2}\right) + y\left(-\ \frac{2\ bd}{a} - \frac{2\ b}{a} i\ \sqrt{af - d^2}\right.$$

$$\left. +\ 2\ e\right) = 0,$$ which reduces to

$$- i\sqrt{af-d^2}\, x - \frac{d}{a}\, i\sqrt{af-d^2} + \frac{af-d^2}{a} - \frac{bd}{a}y - \frac{b}{a}\, i\sqrt{af-d^2}\, y + ey = 0$$

To find P, the point of intersection of the tangents through J_1 and J_2.

Adding

(1.) $$i\sqrt{af-d^2}\, x + \frac{d}{a}\, i\sqrt{af-d^2} + \frac{af-2d^2}{a} - \frac{bd}{a}\, y + \frac{b}{a}\, i\sqrt{af-d^2}\, y + ey = 0$$

$$- i\sqrt{af-d^2}\, x - \frac{d}{a}\, i\sqrt{af-d^2} + \frac{af-2d^2}{a} - \frac{bd}{a}\, y - \frac{b}{a}\, i\sqrt{af-d^2}\, y + ey = 0,$$

we get $\frac{(af-2d^2)}{a} - \frac{bd}{a}y + ey = 0,$

$$\left(\frac{bd}{a} - e\right) y = \frac{af-d^2}{a}.$$

$$\therefore y = \frac{af-d^2}{bd-ae}.$$

Substituting this value of y in (1),

$$i\sqrt{af-d^2}\, x + \frac{d}{a}\, i\sqrt{af-d^2} + \frac{af-2d^2}{a} - \frac{bd}{a}\left(\frac{af-d^2}{bd-ae}\right) + \frac{b}{a}\, i\sqrt{af-d^2}\left(\frac{af-d^2}{db-ae}\right) + e\left(\frac{af-d^2}{bd-ae}\right) = 0,$$

$$a(bd-ae)i\sqrt{af-d^2}\, x + d(bd-ae)i\sqrt{af-d^2} + (af-d^2)(bd-ae) - bd(af-d^2) + b(af-d^2)i\sqrt{af-d^2} + ae(af-d^2) = 0.$$

$$x = -\frac{d(bd-ae) + b(af-d^2)}{a(bd-ae)}.$$

$$P \left[\begin{array}{l} x = \dfrac{de-bf}{bd-ae}, \\[2ex] y = \dfrac{af-d^2}{bd-ae}. \end{array}\right.$$

To find the system of curves when transformation II is applied to

$$ax^2 + 2\,bxy + (c + \lambda)y^2 + 2\,dx + 2\,ey + f = 0.$$

Applying the transformation and dropping primes, we get

$$\frac{a(am^2 - 2\,dlm + fl^2)^2X^2}{[l(am - dl)X - l^2\sqrt{af-d^2}\,y]^2} - \frac{2\,am(am^2 - 2\,dlm + fl^2)X}{l^2(am - dl)X - l^3\sqrt{af-d^2}\,y}$$

$$+ \frac{am^2}{l^2} + \frac{2\,b(am^2 - 2\,dlm + fl^2)^2X}{l[(am - dl)X - l\sqrt{af-d^2}\,y]^2} - \frac{2\,bm(am^2 - 2\,dlm + fl^2)}{l(am - dl)X - l^2\sqrt{af-d^2}\,y}$$

$$+ \frac{(c + \lambda)(am^2 - 2\,dlm + fl^2)^2}{[(am - dl)X - l\sqrt{af-d^2}\,y]^2} + \frac{2\,d(am^2 - 2\,dlm + fl^2)\,X}{l(am - dl)X - l^2\sqrt{af-d^2}\,y}$$

$$- \frac{2\,dm}{l} + \frac{2\,e(am^2 - 2\,dlm + fl^2)}{(am - dl)\,X - l\sqrt{af-d^2}\,y} + f = 0.$$

This may be reduced to the form $\big[a^2m^2l^3f - am^2d^2l^3 - 2\,adml^4f + 2\,d^3ml^4 + af^2l^5 - d^2fl^5\big]X^2 + \big[a^2m^2l^3f - am^2d^2l^3 - 2\,adml^4f + 2\,d^3ml^4 + af^2l^5 - d^2fl^5\big]y^2 + \big[2\,abl^4m^2f - 2\,abdl^3m^3 + 4\,bd^2l^4m^2 - 6\,bdl^5mf + 2\,bf^2l^6 + 2\,a^2el^3m^3 - 6\,adel^4m^2 + 2\,aefl^5m + 4\,d^2el^5m - 2\,defl^6\big]X + \big[2\,abl^3m^3 - 4\,bdl^4m^2 + 2\,bfl^5m - 2\,ael^4m^2 + 4\,del^5m - 2\,efl^6\big]\sqrt{af-d^2}\;y + (c + \lambda)\big[a^2l^3m^4 - 4\,adl^4m^3 + 2\,afl^5m^2 + 4\,d^2l^5m^2 - 4\,dfl^6m + f^2l^7\big] = 0$, which is the equation of a system of concentric circles for each possible transformation.

We find the center of the system to be:-

$$\alpha = -\tfrac{1}{2}\,D = -\tfrac{1}{2}\;\frac{2(2m^2l^3 - 2\,dml^4 + fl^5)(lbf - bdm + mae - del)}{(am^2l^3 - 2\,dml^4 + fl^5)(af - d^2)},$$

$$= \frac{lde - lbf + mbd - mae}{af - d^2}.$$

$$\beta = -\tfrac{1}{2}E = -\tfrac{1}{2}\,\frac{2(am^2l^3 - 2\,dml^4 + fl^5)(mb - le)\sqrt{af-d^2}}{(am^2l^3 - 2\,dml^4 + fl^5)(af - d^2)},$$

$$= \frac{le - mb}{\sqrt{af - d^2}}.$$

To find P', which is the point P transformed by the transformation I.

$$P \begin{cases} x = \dfrac{de - bf}{bd - ae}, \\[2ex] y = \dfrac{af - d^2}{bd - ae}. \end{cases}$$

$$I \begin{cases} x' = \dfrac{lx + m}{y}, \\[2ex] y' = \dfrac{(am - dl)X + (dm - fl)}{\sqrt{af - d^2}\;\;y}. \end{cases}$$

Therefore the coordinates of P' are

$$x' = \frac{\dfrac{lde - lbf + mbd - mae}{bd - ae}}{\dfrac{af - d^2}{bd - ae}}$$

$$x' = \frac{lde - lbf + lbd - mae}{af - d^2}.$$

$$y' = \frac{\dfrac{(am - dk)(de - bf) + (dm - fk)(bd - ae)}{bd - ae}}{\dfrac{\sqrt{af - d^2}\;(af - d^2)}{bd - ae}}$$

$$y' = \frac{le - mb}{\sqrt{af - d^2}}.$$

$$\therefore P' \left[\begin{array}{l} x' = \dfrac{lde - lbf + mbd - mae}{af - d^2} \\[2ex] y' = \dfrac{le - mb}{\sqrt{af - d^2}}, \end{array} \right.$$

which is the point (α, β), the center of the concentric circles.

To find the equations of the lines into which the tangents J_1 P and J_2 P are transformed by the transformation II.

The equation of the tangent J_1P is:-

$$i\sqrt{af-d^2}\, x + \frac{d}{a} i\sqrt{af-d^2} + \frac{(af - d^2)}{a} - \frac{bd}{a} y + \frac{b}{a} i\sqrt{af-d^2}\, y + ly = 0$$

Substituting the transformation II,

$$\frac{i\sqrt{af-d^2}\,(am^2 - 2\,dlm + fl^2)x'}{l(am - dl)\,x' - l^2\sqrt{af-d^2}\,y'} - \frac{mi\sqrt{af-d^2}}{l} + \frac{d}{a} i\sqrt{af-d^2}$$
$$+ \frac{(af - d^2)}{a} + \left(-\frac{bd}{a} + \frac{b\,i\sqrt{af-d^2}}{a} + l\right)\left(\frac{am^2 - 2\,dlm + fl^2}{(am - dl)x' - l\sqrt{af-d^2}\,y'}\right) = 0,$$

which reduces to:--

$$\left[i\sqrt{af-d^2}\,(afl^2 - d^2l^2) + a^2flm - afdl^2 - ad^2lm + d^3l^2\right] x'$$
$$+ \left[i\,(a^2mlf - amld^2 - adl^2f + d^3l^2) - (al^2f - l^2d^2)\sqrt{af-d^2}\right] y'$$
$$+ i\sqrt{af-d^2}(ablm^2 - 2\,bdl^2m + bfl^3) - abdlm^2 + 2bd^2l^2m - bdfl^3$$
$$+ a^2elm^2 - 2\,aedl^2m + aefl^3 = 0$$

, which is the equation of the line into which J_1 P is transformed by II.

On substituting the coordinates of I_1 in this equation, the equation vanishes. Therefore this line passes through I_1.

Also on substituting the coordinates of P' in the equation, it vanishes, and therefore the line also passes through P'.

The equation of the tangent J_2 P is:-

$$- i\sqrt{af-d^2}\, x - \frac{d}{a} i\sqrt{af-d^2} + \frac{af - d^2}{a} - \frac{bd}{a} y - \frac{b}{a} i\sqrt{af-d^2}\, y + ey = 0.$$

Substituting the transformation II,

$$\frac{-i \;\; af-d^2(am^2 - dlm + fl^2)x'}{l(am - dl)x' - l^2\sqrt{af-d^2}\,y'} + \frac{mi\sqrt{af-d^2}}{l} - \frac{d}{a}i\sqrt{af-d^2}$$

$$+ \frac{af - d^2}{a} + \left(-\frac{bd}{a} - \frac{b}{a}i\sqrt{af-d^2} + e\right)\left(\frac{am^2 - 2alm + fl^2}{(am - dl)x' - l\sqrt{af-d^2}\,y'}\right) = 0$$

which reduces to

$$\left[i\sqrt{af-d^2}(d^2l^2 - afl^2) + a^2lmf - afdl^2 - ad^2lm + d^3l^2\right]x'$$
$$+ \left[i\,(ad^2ml - a^2lmf + afdl^2 - d^3l^2) - (afl^2 - d^2l^2)\sqrt{af-d^2}\right]y'$$
$$- i\sqrt{af-d^2}\,(albm^2 - 2bdl^2m + bfl^3) - abdm^2l + 2bd^2l^2m - bdfl^3 + a^2elm^2$$
$$- 2aedl^2m + aefl^3 = 0,$$

which is the equation of the line into which J_2P is transformed by II.

Since the co-ordinates of I_2 and also those of P' satisfy this equation, the line passes through the points I_2 and P'.

The results obtained may be stated in the following Theorems.

There are ∞^4 transformations which transform two conjugate imaginary points into the circular points at infinity. A conic passing through the two conjugate imaginary points, being transformed by any one of these transformations, is transformed into a circle.

A system of conics doubly tangent at two conjugate imaginary points, J_1 and J_2, is transformed into a system of concentric circles, by any one of the transformations which transform the conjugate imaginary points into the two circular points at infinity. The circles of a concentric system are therefore doubly tangent at the circular points. The point of intersection of the two tangents to the system of conics at the two conjugate imaginary points is transformed by the same

transformation into the center of the concentric circles and the tangents, themselves, into two lines passing through the circular points and intersecting at the center of the circles.

4. The Relation of the Foci of Conics and the Circular Points.

From an arbitrary point, P, in a plane, a certain number of tangents may be drawn to a curve in the same plane, which determines the class of the curve. If from I_1 and I_2, the two circular points, two systems of tangents are drawn to the curve, then the foci of the curve are defined as the points of intersection of these two systems.

Therefore, in the case of a general conic, since two tangents may be drawn to it from each point, four foci are obtained as shown in Fig (6).

We wish next to determine the system of conics which is inscribed in the lines joining the foci $F_1(+c, o)$ and $F_2(-c, o)$ each with the two circular points I_1 and I_2,[1] which were defined

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[1]This problem was investigated by M.E.Laguerre, who gave to it the following formulation:-

The equation of all conics having the common focus (α, β) is $(x-\alpha)^2 + (y-\beta)^2 = X^2$, in which X is a linear function of x and y.

Factoring the above equation,

$$\left[(x-\alpha) + \sqrt{-1}\,(y-\beta)\right]\left[(x-\alpha) - \sqrt{-1}\,(y-\beta)\right] = X^2.$$

From this form, it follows that these conics are tangent to the two straight lines

$$(x-\alpha) + \sqrt{-1}\,(y-\beta) = 0$$

and $$(x-\alpha) - \sqrt{-1}\,(y-\beta) = 0.$$

Nouvelles Annales de Mathematiques, 1852.
Oeuvres de Laguerre, Vol, II. p.1, Paris 1905.

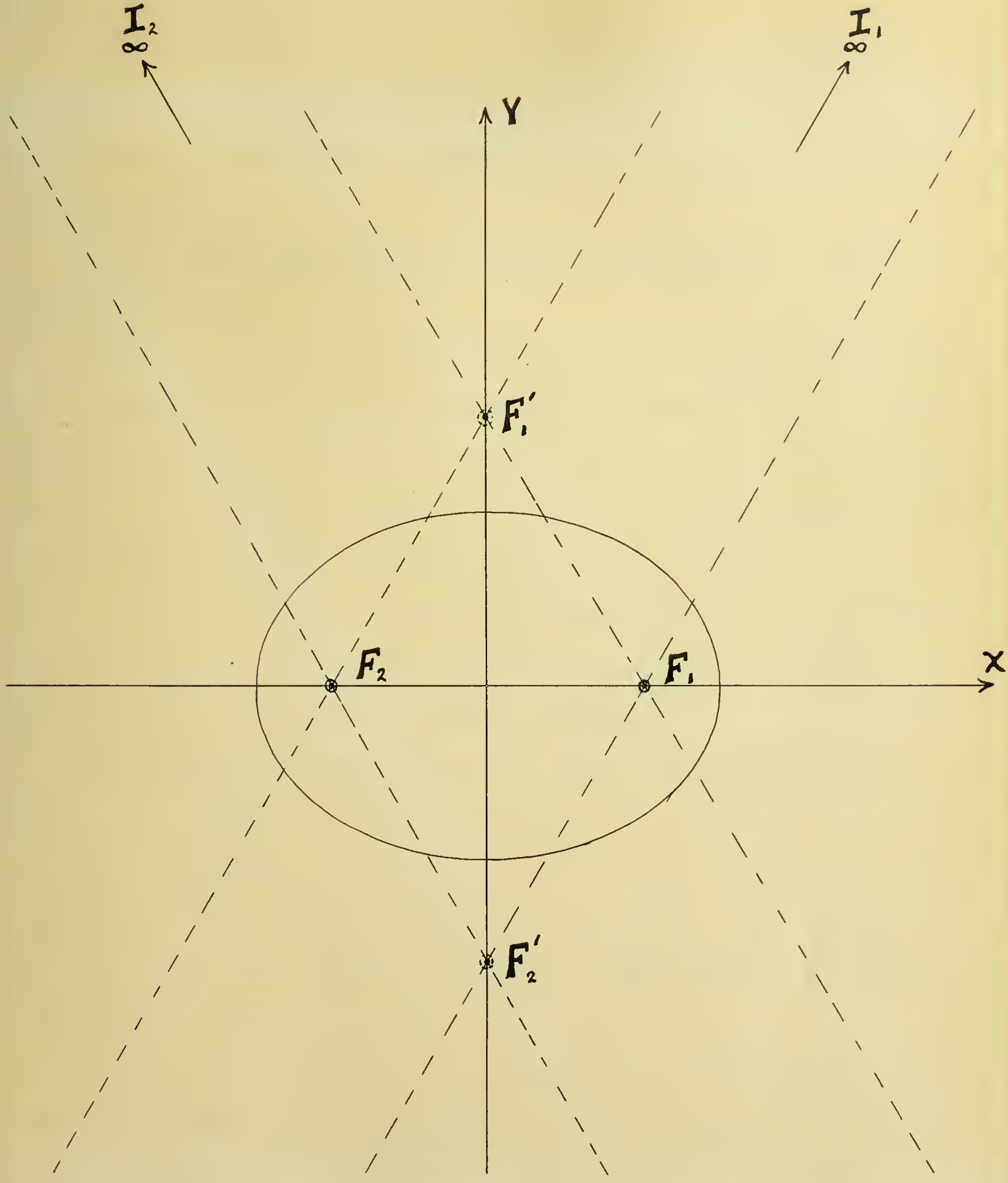

Fig. 6.

by Laguerre as the Isotropic Lines.[1]

Since the slope of two of these lines is +i and that of the other two -i and lines with opposite slopes intersect at F_1 and F_2, these lines form a rhombus. Therefore we must stop to prove the following theorem:

All conics inscribed in a rhombus are coaxial, having the diagonals as axes.

Take any point T as the point of tangency on b_1, Fig. (7). Then by means of the Brianchon Theorem, we can construct the inscribed conic, tangent at this point, but to prove our theorem, we must find the points of tangency of the four given tangents. We can do this by choosing the Brianchon Point properly.

1. To determine the point of tangency of e_1. We will take the five tangents as in the figure, a_1, b_1, c_1, d_1, and e_1. We wish to find f_1, such that e_1 and f_1 coincide.

The Brianchon Point must lie on the line connecting a_1 b_1, and d_1 e_1. Therefore we will take B_1 at the center of the rhombus. Then completing the construction we find e_1 and f_1 coincide. But the line drawn through b_1 c_1, and B_1 passes thru e_1 f_1. However since these two tangents coincide, this point of intersection is the point of tangency, which is T_1.

2. In a similar way, by choosing the tangents a_2, b_2, c_2, d_2, and e_2 as in the figure and the Brianchon Point B_2 at infinity on the line through a_2 b_2 and d_2 e_2, we find that the tangents e_2 and f_2 coincide and the point of tangency is T_2.

3. Also, by choosing the tangents a_3, b_3, c_3, d_3, and e_3

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[1]Oeuvres de Laguerre, Vol.II. p. 88.

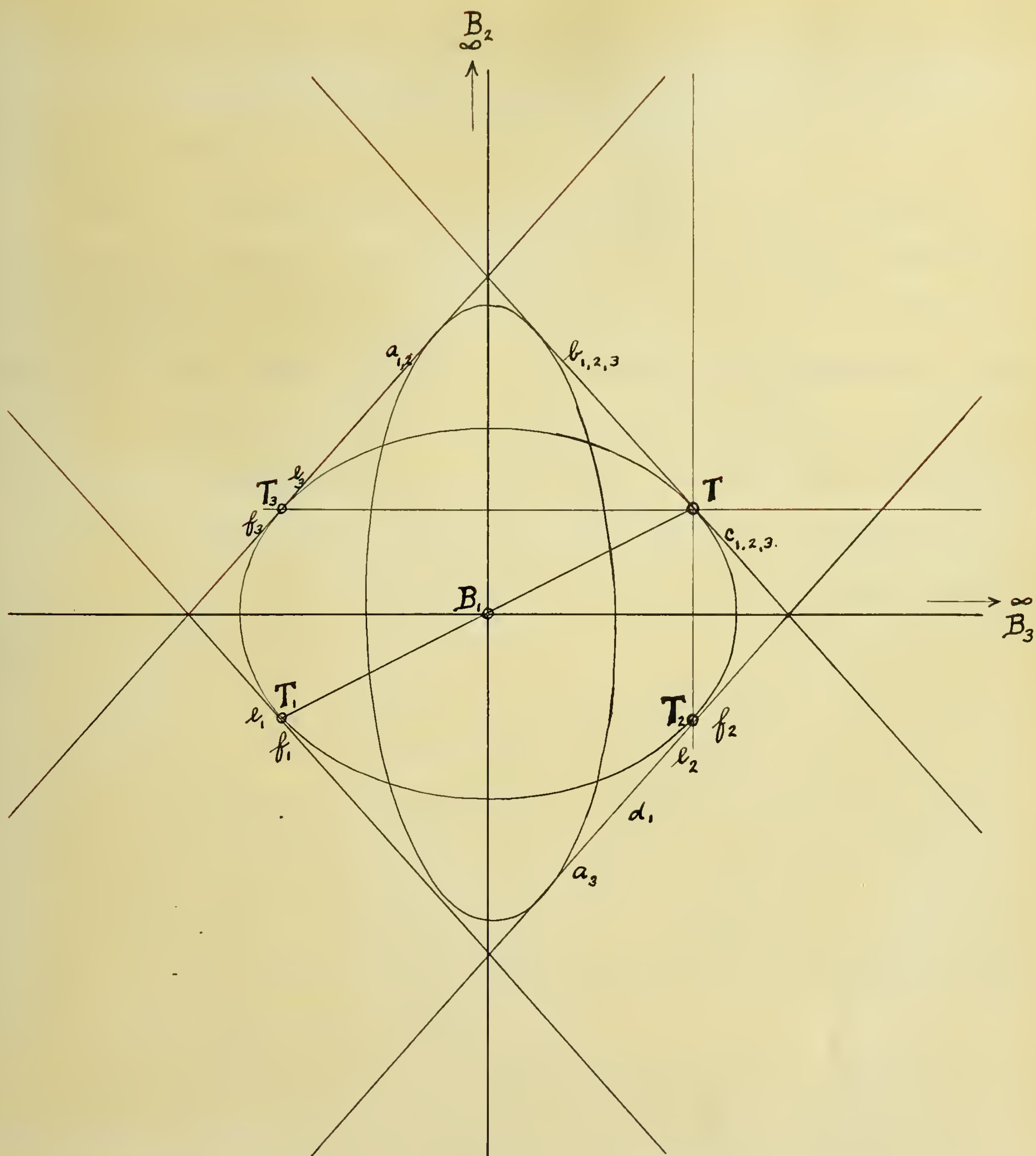

Fig. 7.

as in the figure and B_3 at infinity on the line through a_3 b_3 and d_3 e_3, we find that the tangents e_3 and f_3 coincide and the point of tangency is T_3.

From the construction, we can easily see that the lines drawn through opposite points of tangency are conjugate diameters and intersect at the center of the rhombus. Also these diameters make equal angles with each of the diagonals of the rhombus. Therefore, these diagonals must be conjugate diameters of the conic, and since they are perpendicular to each other are the axes of the conic, for there can be only one pair of rectangular conjugate diameters of a conic, unless all of them are rectangular, when any pair may be called the axes.

We have therefore shown that the diagonals of the rhombus are the axes of one of the inscribed conics. But T was taken as any point of tangency on the tangent b. Now, for every different position of T a different conic will be constructed, but every one will be constructed in the same way as the one above and will therefore have the same axes as the above, namely the diagonals of the rhombus, which was to be proved.

Therefore the system of conics inscribed in the isotropic lines passing through the foci F_1 and F_2 is coaxial, having the coordinate axes as its axes.

We will now find the equation of this system of conics.

The equations of the given lines are:-

Line $F_1 I_2 \equiv y + i(x - c) = 0$ (1).

" $F_1 I_1 \equiv y - i(x - c) = 0$ (2).

" $F_2 I_2 \equiv y + i(x + c) = 0$ (3).

" $F_2 I_1 \equiv y - i(x + c) = 0$ (4).

Their points of intersection are

$F_1\ (+c\ ,\ o)$, $F_2(-c\ ,\ o)$, $F_1'\ (o\ ,\ ic)$, and $F_2'\ (o, -ic)$.

If we take any six intersecting lines, as

$$p_1 = a_1x + b_1y + c_1 = 0$$
$$p_2 = a_2x + b_2y + c_2 = 0$$
$$q_1 = d_1x + e_1y + f_1 = 0$$
$$q_2 = d_2x + e_2y + f_2 = 0$$
$$r = \alpha_1x + \beta_1y + \gamma = 0$$
$$s = \alpha_2x + \beta_2y + \gamma = 0,$$
$$\text{then } p_1p_2 - \lambda r^2 = 0 \qquad (1.)$$

is the equation of a system of conics, all of which are doubly tangent at the point of intersection of $r^2 = 0$ and $p_1 = 0$ and that of $r^2 = 0$ and $p_2 = 0$ Fig. (8).

Also $q_1\ q_2 - \mu s^2 = 0$ (2.)

is the equation of a system of conics all of which are doubly tangent at the point of intersection of $s^2 = 0$ and $q_1 = 0$ and that of $s^2 = 0$ and $q_2 = 0$.

However, if (1) and (2) represent the same system of conics $p_1p_2 - \lambda r^2$ is identically equal to $q_1q_2 - \mu s^2$.

$$\therefore\ p_1\ p_2 - q_1\ q_2 \equiv \lambda r^2 - \mu s^2. \qquad (3).$$

This is the condition which must exist for the system of conics inscribed in the given rhombus.

$$\text{Let } p_1 \equiv F_1I_1 \equiv y - i(x - c) = 0$$
$$p_2 \equiv F_2I_1 \equiv y - i(x + c) = 0$$
$$q_1 \equiv F_1I_2 \equiv y + i\ (x - c) = 0$$
$$q_2 \equiv F_2I_2 \equiv y + i(x + c) = 0$$
$$r \equiv \alpha_1x + \beta_1y + \gamma_1 = 0$$
$$s \equiv \alpha_2x + \beta_2y + \gamma_2 = 0$$

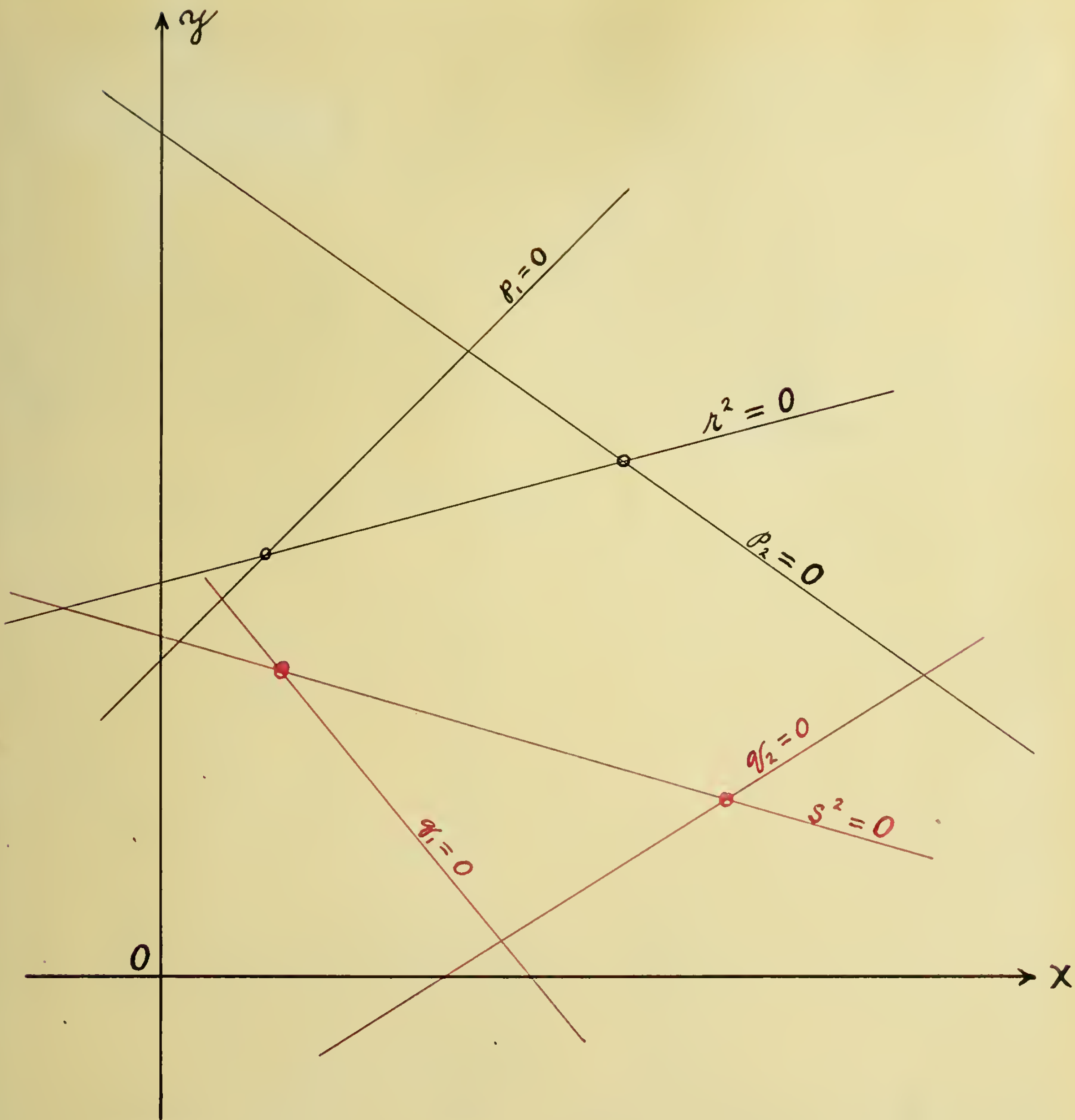

Fig. 8.

Substituting these values in (3.),

$[y - i(x - c)][y - i(x + c)] - [y + i(x - c)][y + i(x + c)] = \lambda r^2 - \mu s^2.$ $y^2 - i\,y(x - c) - i\,y(x + c) - x^2 + c^2 - y^2 - i\,y(x - c) - i\,y(x + c) + x^2 - c^2 = \lambda r^2 - \mu s^2.$

$- i\,xy + i\,cy - i\,xy - i\,cy - i\,xy + i\,cy - i\,xy - i\,cy = \lambda r^2 - \mu s^2.$

$- 4\,i\,xy = \lambda(\alpha_1 x + \beta_1 y + \gamma_1)^2 - \mu(\alpha_2 x + \beta_2 y + \gamma_2)^2.$ $-4ixy$
$= \lambda\alpha_1^2 x^2 + 2\lambda\alpha_1\beta_1 xy + 2\lambda\alpha_1\gamma_1 x + \lambda\beta_1^2 y^2 + 2\lambda\beta_1\gamma_1 y + \lambda\gamma_1^2$
$- \mu\alpha_2^2 x^2 - 2\mu\alpha_2\beta_2 xy - 2\mu\alpha_2\gamma_2 x - \mu\beta_2^2 y^2 - 2\mu\beta_2\gamma_2 y - \mu\gamma_2^2$

Equating coefficients of like terms,

$\lambda\alpha_1^2 - \mu\alpha_2^2 = 0.$ (1,)

$\lambda\alpha_1\beta_1 - \mu\alpha_2\beta_2 = -2i.$ (2.)

$\lambda\alpha_1\gamma_1 - \mu\alpha_2\gamma_2 = 0.$ (3.)

$\lambda\beta_1^2 - \mu\beta_2^2 = 0.$ (4.)

$\lambda\beta_1\gamma_1 - \mu\beta_2\gamma_2 = 0.$ (5.)

$\lambda\gamma_1^2 - \mu\gamma_2^2 = 0.$ (6.)

From (1.) $\alpha_1 = \pm\alpha_2\sqrt{\frac{\mu}{\lambda}}.$ (7.)

" (4.) $\beta_1 = \pm\beta_2\sqrt{\frac{\mu}{\lambda}}.$ (8.)

" (6.) $\gamma_1 = \pm\gamma_2\sqrt{\frac{\mu}{\lambda}}.$ (9.)

Substituting the values of α_1 and β_1 as given in (7.) and (8.) in (2.), we find that α_1 and β_1 must be opposite in sign.

Therefore let $\alpha_1 = +\alpha_2\sqrt{\frac{\mu}{\lambda}}$ (10.)

and $\beta_1 = -\beta_2\sqrt{\frac{\mu}{\lambda}}.$ (11.)

Substituting the values of α_1 and β_1 as given in (10.) and (11.) in (2.),

$$- \lambda\alpha_2\beta_2\frac{\mu}{\lambda} - \mu\alpha_2\beta_2 = -2i.$$

$$2\mu\alpha_2\beta_2 = +2i$$

and $\alpha_2 = \dfrac{i}{\mu\beta_2}.$ (12.)

From (10.) $\dfrac{\alpha_2}{\alpha_1} = \sqrt{\dfrac{\lambda}{\mu}}$.

From (11.) $- \dfrac{\beta_2}{\beta_1} = \sqrt{\dfrac{\lambda}{\mu}}$.

$\therefore \quad \dfrac{\alpha_2}{\alpha_1} = - \dfrac{\beta_2}{\beta_1}$,

and $\alpha_2 \beta_1 + \alpha_1 \beta_2 = 0$. (13.)

Multiplying (3.) by β_2, $\lambda \alpha_1 \beta_2 \gamma_1 - \mu \alpha_2 \beta_2 \gamma_2 = 0$.

" (5.) by α_2, $\lambda \alpha_2 \beta_1 \gamma_1 - \mu \alpha_2 \beta_2 \gamma_2 = 0$.

$\gamma_1 (\alpha_1 \beta_2 - \alpha_2 \beta_1) = 0$.

From (13.), we see that $\alpha_1 \beta_2 - \alpha_2 \beta_1 \neq 0$.

$\therefore \gamma_1 = 0$.

and from (9.) $\gamma_2 = 0$.

We still have six unknown numbers and only three equations by which to determine them, (1,), (2.), and (4.). Since there are therefore an infinite number of solutions, we may let $\lambda = \mu$.

Then from (10.), $\alpha_1 = \alpha_2$, and from (11.), $\beta_1 = - \beta_2$.

From (12.), $\alpha_2 = \dfrac{i}{\lambda \beta_2}$.

$\therefore \quad \alpha_1 = \dfrac{i}{\lambda \beta_2}$.

We have therefore $\gamma_1 = 0$; $\gamma_2 = 0$; $\alpha_1 = \dfrac{i}{\lambda \beta_2}$; $\alpha_2 = \dfrac{i}{\lambda \beta_2}$; and $\beta_1 = - \beta_2$.

Substituting these values in

$$[y - i(x - c)] [y - i(x + c)] - \lambda r^2 = 0,$$

We get

$$y^2 - iy(x - c) - iy(x + c) - x^2 + c^2 + \frac{\lambda x^2}{\lambda^2 \beta_2^2} + \frac{2 \lambda i \beta_2}{\lambda \beta_2} xy$$

$$- \lambda \beta_2^2 y^2 = 0$$

$$y^2 - i\,xy + i\,cy - i\,xy - i\,cy - x^2 + c^2 + \frac{x^2}{\lambda\beta_2^2} + 2\,i\,xy - \lambda\beta_2^2 y^2 = 0,$$

$$y^2 - x^2 + c^2 + \frac{x^2}{\lambda\beta_2^2} - \lambda\beta_2^2 y^2 = 0.$$

$$\lambda\beta_2^2\, y^2 - \lambda\beta_2^2\, x^2 + \lambda\beta_2^2\, c^2 + x^2 - \lambda^2\beta_2^4\, y^2 = 0,$$

$$(1 - \lambda\beta_2^2)\, x^2 + \lambda\beta_2^2(1 - \lambda\beta_2^2)\, y^2 + \lambda\beta_2^2\, c^2 = 0, \quad \text{I.}$$

which is the equation of the required system of conics.

Discussion of Equation I.

$$(1 + \lambda\beta_2^2)\, x^2 + \lambda\beta_2^2\,(1 - \lambda\beta_2^2)\, y^2 + \lambda\beta_2^2\, c^2 = 0, \qquad (1.)$$

$$\text{or} \quad \frac{1 - \lambda\beta_2^2}{\lambda\beta_2^2\, c^2}\, x^2 + \frac{1 - \lambda\beta_2^2}{c^2} \quad + 1 = 0. \qquad (2.)$$

k = any positive number.

1. Let $\lambda\beta_2^2 = -\,k$.

From (2.), $\dfrac{1 + k}{-\,kc^2}\, x^2 + \dfrac{1 + k}{c^2}\, y^2 + 1 = 0,$

$$\text{or} \quad \frac{1 + k}{kc^2} x2 - \frac{1 + k}{c^2} = 1,$$

which is the equation of a system of confocal hyperbolas, with foci at F_1 $(c\,, o)$ and F_2 $(-c\,, o)$ as shown in Fig. (9).

2. Let $\lambda\beta_2^2 = 0$.

From (1.), $x^2 = 0$, which is the degenerate conic, the y - axis is taken twice.

3. Let $\lambda\beta_2^2 = k$, where $0 < k < 1$.

From (2.), $\dfrac{1 - k}{kc^2}\, x^2 + \dfrac{1 - k}{c^2}\, y^2 + 1 = 0,$

which is a system of imaginary conics with foci F_1' $(o\,, ic)$ and $F'_2(o\,, -ic)$.

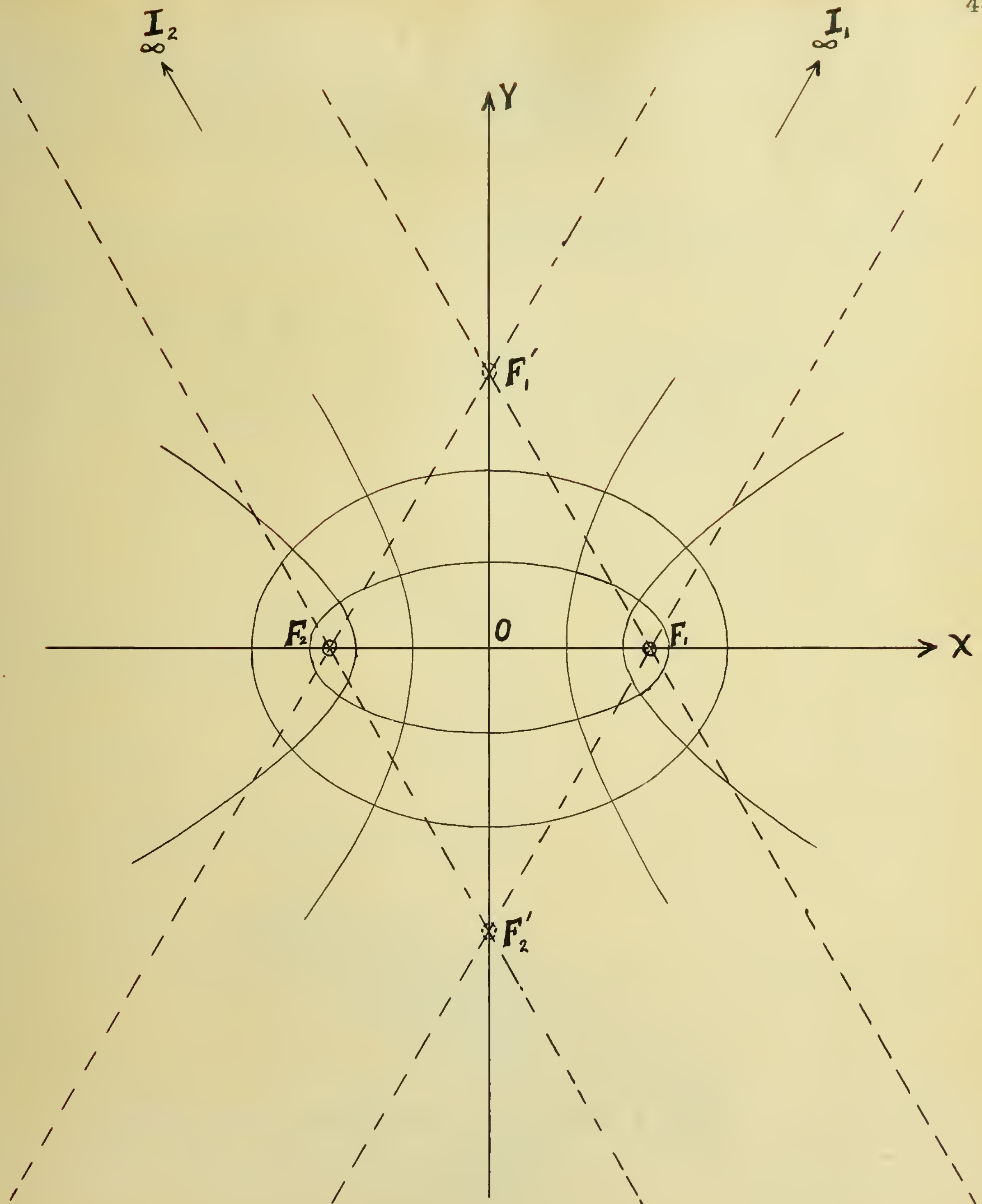

Fig. 9.

4. Let $\lambda\beta_2^2 = 1$.

From $x^2 + \lambda\beta_2^2 y^2 + \dfrac{\lambda\beta_2^2 c^2}{1-\lambda\beta_2^2} = 0$,

If $\lambda\beta_2^2$ approaches 1 in the ascending order of the number system,
$$x^2 + y^2 + \infty = 0,$$
or $x^2 + y^2 = (i\infty)^2$, which is an imaginary conic.

If $\lambda\beta_2^2$ approaches 1 in the descending order,
$$x^2 + y^2 - \infty = 0,$$
or $x^2 + y^2 = \infty^2$, which is a circle with a radius of infinite length and center at the origin.

5. Let $\lambda\beta_2^2 = k > 1$.

$$\frac{1-k}{kc^2}x^2 + \frac{1-k}{c^2}y^2 + 1 = 0.$$

Since $1 - k$ is negative,
$$-\frac{k-1}{kc^2}x^2 - \frac{k-1}{c^2}y^2 + 1 = 0$$
or $\dfrac{k-1}{kc^2}x^2 + \dfrac{k-1}{c^2}y^2 = 1$,
which is the equation of a system of confocal ellipses with foci at F_1 (c,o) and F_2 (-c,o) Fig. (9).

<u>To find the equations of $r = 0$ and $s = 0$.</u>

$$r \equiv \alpha_1 x + \beta_1 y + \gamma_1 = 0.$$
$$s \equiv \alpha_2 x + \beta_2 y + \gamma_2 = 0.$$

Substituting the following values $\gamma_1 = 0$; $\gamma_2 = 0$; $\alpha_1 = \dfrac{i}{\lambda\beta_2}$; $\alpha_2 = \dfrac{i}{\lambda\beta_2}$; and $\beta_1 = -\beta_2$,

$$r \equiv \frac{i}{\lambda\beta_2}x - \beta_2 y = 0, \quad \text{or } y = \frac{i}{\lambda\beta_2^2}x.$$
$$s \equiv \frac{i}{\lambda\beta_2}x + \beta_2 y = 0, \quad \text{or } y = -\frac{i}{\lambda\beta_2^2}x.$$

Therefore r = 0 and s = 0 are imaginary lines with equal but opposite slopes, passing thru the center of the rhombus.

For each value of $\lambda\beta_2^2$ we have a definite conic inscribed in the rhombus and to each conic is associated two imaginary lines r = 0 and s = 0, which intersect at the center of the conic and pass through the points at which the given isotropic lines are tangent to the conic.

A Discussion of the System of Parabolas, which is tangent to the lines joining their foci to the circular points.

However, we must first prove the following theorem:-

The system of conics, which have three common tangents, such that the line which bisects the angle formed by the first two is perpendicular to the third at the point of tangency, have a common axis, which is the above bisector.

Given the two tangents a and b which intersect at A; Fig. (10), also two coincident tangents d and e, which are perpendicular to Ade, the bisector of angle ab, at their point of tangency de.

To prove that the system of conics tangent to d and e at de and also to a and ab has the line Ade as a common axis.

Take any other tangent c, cutting the line Ade at C. Now we have given five elements, which determine a conic. By means of the Brianchon Theorem any other tangent f to the conic can be constructed. We know that the Brianchon point lies on Ade, for this line connects the opposite vertices ab and de.

In order to prove the theorem we must show that the tangent f can be drawn so that it passes through C and makes an angle with Ade, which is equal to that which c makes with Ade, but is opposite in sign. Therefore, to choose the Brianchon

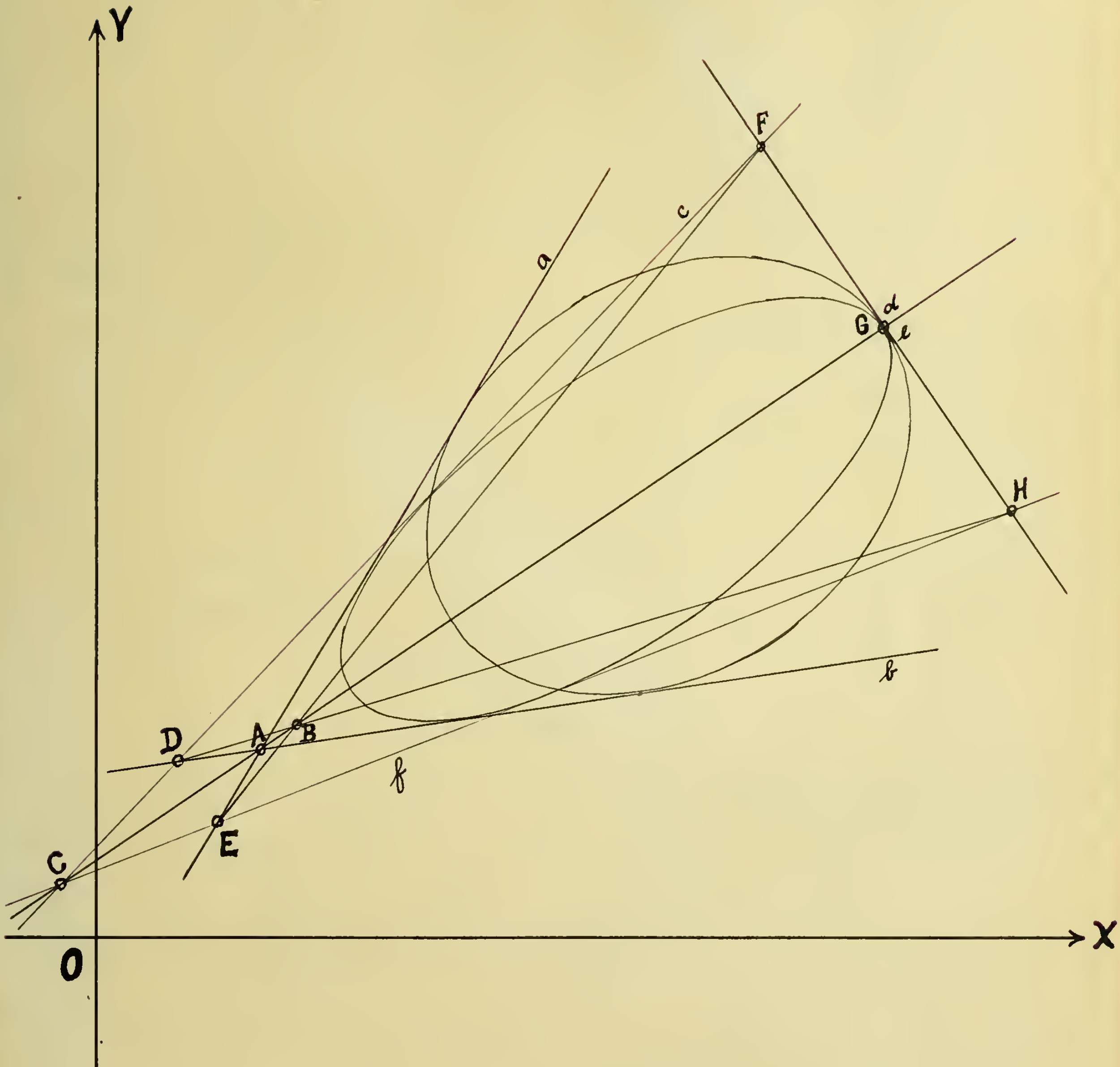

Fig. 10.

Point properly, lay off the distance AE on a equal to AD on b. We will now take E as the intersection of a and f. Draw the line connecting af and cd, and its point of intersection with Ade is the Brianchon Point, B. Draw a line from be thru B, until it intersects e. This point of intersection is the point ef. Draw the tangent f thru E and ef.

We must now prove that the tangent f, constructed in this way, passes through C and makes the required angle with Ade.

In the figure, designate the intersections as follows, bc by D, af by E, cd by F, de by G, and ef by H.

By hypothesis, $\angle aAB = \angle bAB$, and

by construction, $AE = AD$,

$\therefore$ since $\triangle ABD = \triangle ABE$,

$\angle ABD = \angle ABE$.

Therefore, since EF and DH make equal angles with AG, the lines FD and HE must meet in a common point, C, on AG and make angles with AG, which are equal but opposite in sign.

Therefore the line AG is the axis of the conic determined by the tangents a, b, c, d, e, and f, for each pair of these tangents (a and b), (c and f), and (d and e) intersect in common points on AG and make equal angles with it.

But c is any line which determines any conic of the original system. So by taking c in every possible position, we can determine the corresponding tangent, f, for each conic in the same way, and having the same properties as the one demonstrated, and every conic of the system is determined.

Therefore every conic of the system has the line AG as an axis. Q. E. D.

To find the system of conics which is tangent to a, b, d, and e, when a and b are two lines intersecting the x- axis in the common point C (k,0) and having equal but opposite slopes, and d and e are coincident and are perpendicular to the x- axis at infinity. Fig. (11.)

Since the system of conics has the x- axis as the common axis, the general equation is:

$$ax^2 + by^2 + 2\,dx + e = 0$$

Since de is at infinity, the system passes through the point $(x = \infty,\ y = 0,\ \frac{y}{x} = 0)$. Substituting these values in

$$a + b\frac{y^2}{x^2} + \frac{2d}{x} + \frac{e}{x^2} = 0,$$

We get $\quad a = 0.$

$\therefore$ the general equation reduces to

$by^2 + 2\,dx + e = 0 \quad$ or $y^2 = -\frac{2d}{b}x - \frac{e}{b}$, which is the equation of a system of parabolas and which may be put in the form

(1.) $\quad y^2 = 4p(x - \alpha)$, where α is the intercept on the x- axis and 4p the latus rectum.

Take the equations of a and b as

(2.) $\quad a \equiv y - mx + mk = 0$

(3.) $\quad b \equiv y + mx - mk = 0,$

in which m and k are given.

To find the value of p, solve (1.) and (2.) simultaneously.

$y - mx + mk = 0 \qquad$ (2.)

$y^2 - 4px + 4p\alpha = 0. \qquad$ (1.)

From (2.), $\quad x = \frac{y + mk}{m}.$

Substituting this value in (1.),

$$y^2 - \frac{4py}{m} - 4pk + 4p\alpha = 0.$$

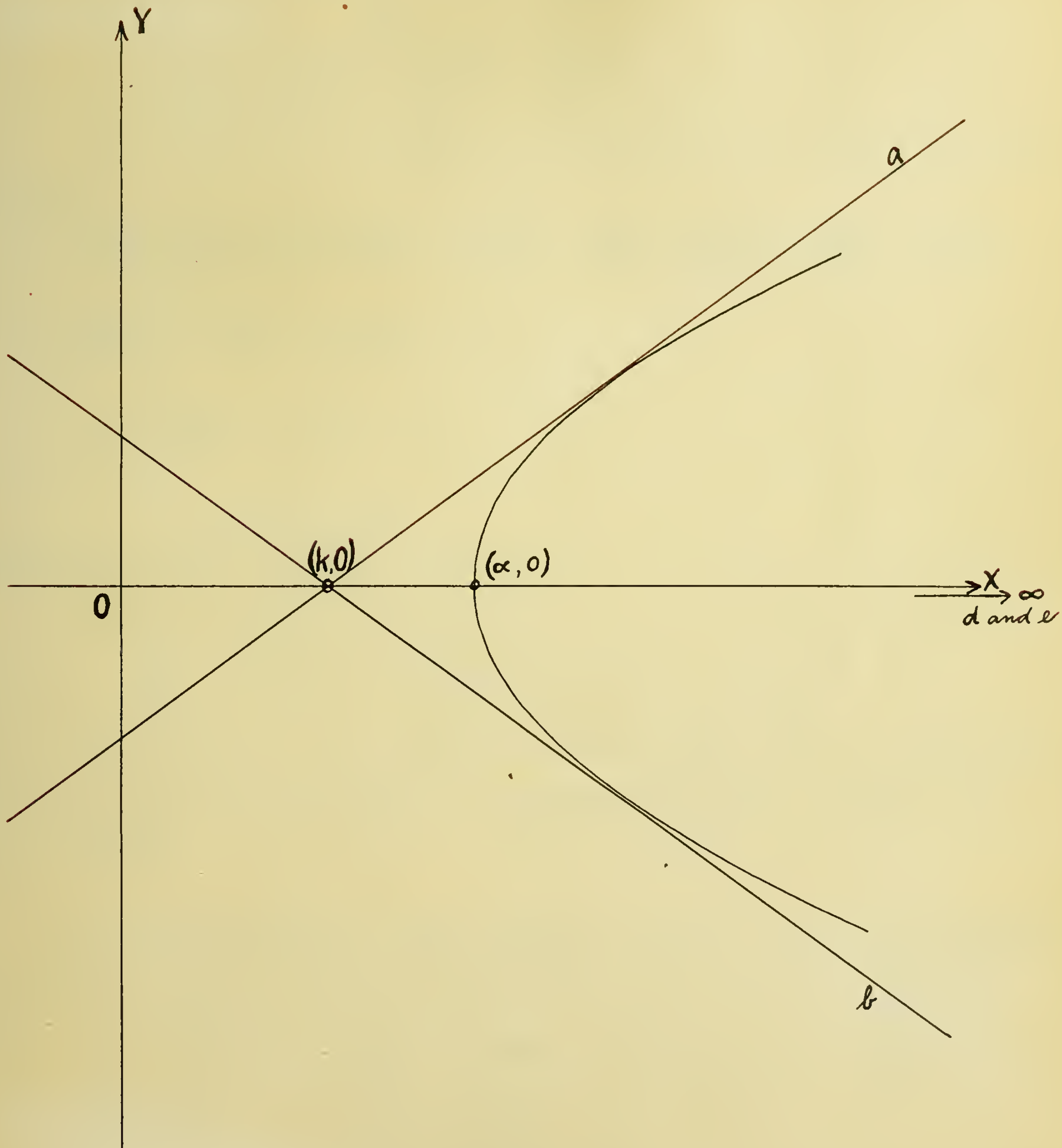

Fig. 11.

$$y = \frac{\frac{4p}{m} \pm \sqrt{\frac{16p^2}{m^2} + 16pk - 16p\alpha}}{2},$$

$$y = \frac{2p}{m} \pm \frac{2}{m}\sqrt{p^2 + pkm^2 - p\alpha m^2}.$$

But since (2.) is tangent to (1.), the two values of y must be equal.

$$\therefore \frac{2p}{m} + \frac{2}{m}\sqrt{p^2 + pkm^2 - p\alpha m^2} = \frac{2p}{m} - \frac{2}{m}\sqrt{p^2 + pkm^2 - p\alpha m^2}.$$

$$\frac{4}{m}\sqrt{p^2 + pkm^2 - p\alpha m^2} = 0,$$

$$p + km^2 - \alpha m^2 = 0,$$

$$p = m^2(\alpha - k).$$

Substituting this value of p in (1.),

$$\therefore y^2 = 4m^2(\alpha - k)(x - \alpha),$$

(4.) $$y^2 = (4m^2\alpha - 4m^2k)x - 4m^2\alpha^2 + 4m^2\alpha k,$$

which is the equation of a system of parabolas, with $y = 0$ as the common axis and all of which are tangent to a and b, where α is the parameter.

By substituting for the tangents a and b, the lines joining F_1 (c,o) and $F_2(-c,0)$ with I_1 and I_2, we will have the following cases. A. Tangent to F_1I_1 and F_1I_2, with $y = 0$ as axis.

B. " " F_2I_1 and F_2I_2, with $y = 0$ " " .

C. " " F_1I_2 and F_2I_1, with $x = 0$ " " .

D. " " F_1I_1 and F_2I_2, with $x = 0$ " " .

A. To find the system of parabolas, which is tangent to F_1I_1 and F_1I_2 and has $y = 0$ as the common axis. Fig. 12.

$\therefore$ take $a \equiv F_1I_1 \equiv y - i(x - c) = 0.$ (5.)

and $b \equiv F_1I_2 \equiv y + i(x - c) = 0.$ (6.)

Comparing (5.) with (2.) and (6.) with (3.), we see $m = i$ and $k = c$.

$\therefore$ Substituting these values in (4.),

$$y^2 = (4c - 4\alpha)\,x + 4\alpha^2 - 4\alpha c.$$

$$\text{II.} \qquad y^2 = 4\,(c - \alpha)(x - \alpha),$$

which is the equation of a system of parabolas, with α as the parameter, which is the intercept on the x- axis.

Also $(c - \alpha)$ is the distance of the intersection on the x-axis from the focus. $\therefore$ c is the abscissa of the focus, and the system of parabolas has the common focus F_1 (c, o) and the system is a system of confocal parabolas.

Discussion of Equation II.

$$y^2 = 4(c - \alpha)(x - \alpha).$$

1. When $\alpha < c$.

The equation represents a system of confocal parabolas for $x \geqq \alpha$

When $x < \alpha$, y is imaginary.

2. When $\alpha = c$.

The equation represents a degenerate parabola, the x-axis taken twice, for all values of x.

3. When $\alpha > c$.

The equation represents a system of confocal parabolas for $x \leqq \alpha$.

When $x > \alpha$, y is imaginary.

The focus in each case is F_1 (c,o).

B. To find the system of parabolas, which is tangent to $F_2 I_1$ and $F_2 I_2$ and has $y = 0$ as the common axis.

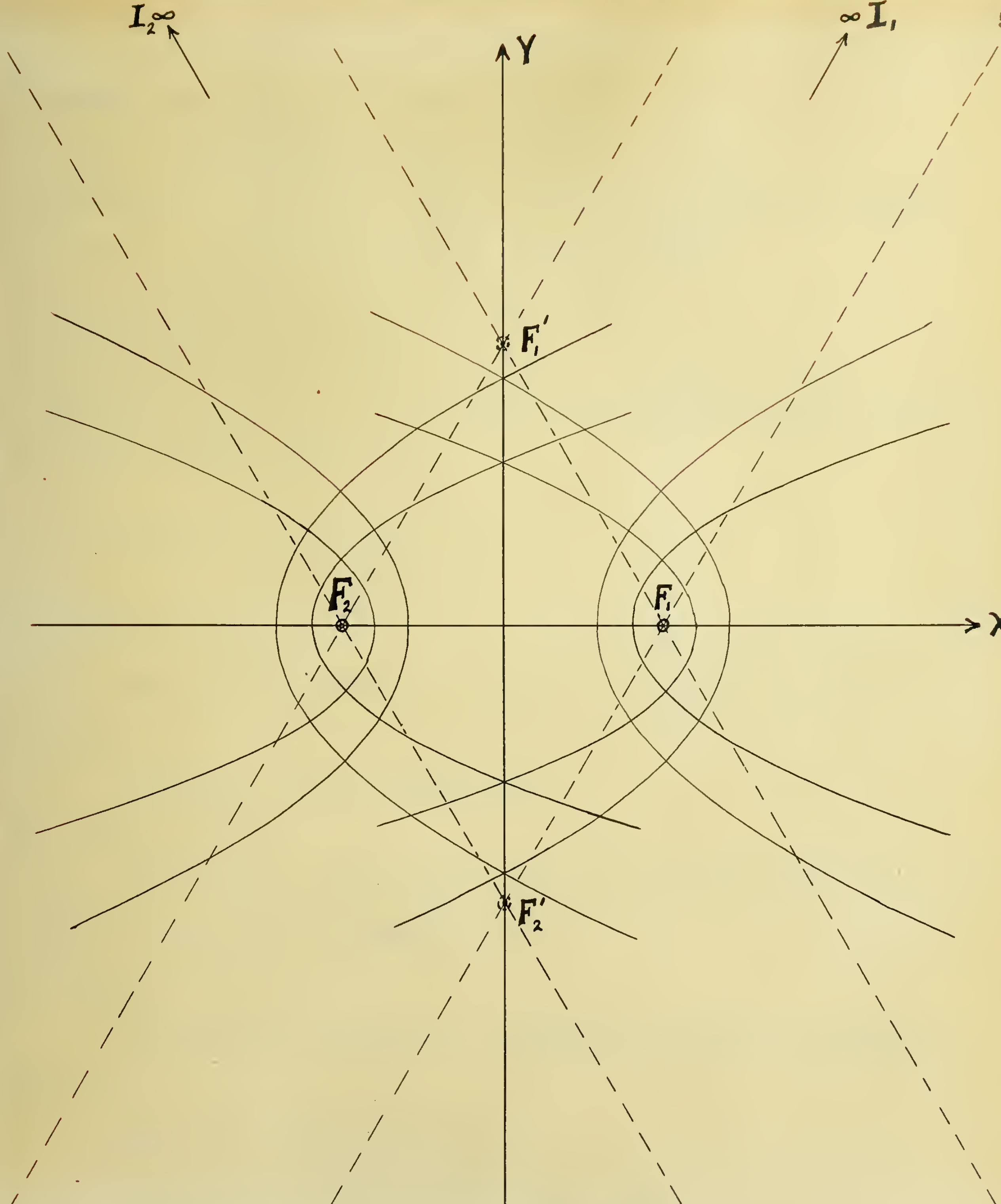

Fig. 12.

$\therefore$ take $a \equiv F_2I_1 \equiv y - i(x + c) = 0$ (7.)

and $b \equiv F_2I_2 \equiv y + i(x + c) = 0$ (8.)

Comparing (7.) with (2.) and (8.) with (3.), we see $m = i$ and $k = -c$. $\therefore$ Substituting these values in (4.),

$$y^2 = (-4c - 4\alpha)x + 4\alpha^2 + 4\alpha c.$$

III $$y^2 = 4(-c - \alpha)(x - \alpha),$$

which is the equation of a system of confocal parabolas, having the common focus $F_2(-c, o)$.

Discussion of Equation III.

$$y^2 = 4(-c - \alpha)(x - \alpha).$$

1. When $\alpha < (-c)$.

The equation represents a system of confocal parabolas for $x \geqq \alpha$.

When $x < \alpha$, y is imaginary.

2. When $\alpha = -c$.

The equation represents the degenerate parabola, the x- axis taken twice, for all values of x.

3. When $\alpha > -c$.

The equation represents a system of confocal parabolas for $x \leqq \alpha$.

When $x > \alpha$, y is imaginary.

The focus in each case is $F_2(-c, o)$.

C. To find the system of parabolas, which is tangent to F_1I_2 and F_2I_1 and has $x = 0$ as the common axis.

The general equation of the system of parabolas with $x = 0$ as the common axis is (9), $x^2 = 4p(y - \beta)$, where β is the intercept on the y- axis and p is the distance of the intersection on the y -

axis from the focus.

(10.) $a \equiv y - mx - k = 0.$

(11.) $b \equiv y + mx - k = 0.$

Solving (9.) and (10.) simultaneously

$$x = 2pm \pm 2\sqrt{p^2m^2 + pk - p\beta}.$$

But since (10.) is tangent to (9.) the two values of x must be equal.

$$\therefore\ 2pm + 2\sqrt{p^2m^2 + pk - p\beta} = 2pm - 2\sqrt{p^2m^2 + pk - p\beta}.$$

$$4\sqrt{p^2m^2 + pk - p\beta} = 0.$$

$$pm^2 + k - \beta = 0.$$

$$p = \frac{\beta - k}{m^2}.$$

Substituting this value of p in (9.),

(12.) $x^2 = 4\,\dfrac{\beta - k}{m^2}(y - \beta).$

Take $a \equiv F_2I_1 \equiv y - ix - ic = 0.$ (13.)

$b \equiv F_1I_2 \equiv y + ix - ic = 0.$ (14.)

Comparing (13.) with (10.) and (14.) with (11.), we see $m = i$ and $k = ic.$ Substituting these values in (12.),

IV $x^2 = 4(ic - \beta)(y - \beta).$

If $\beta = ic$,

The equation represents the real degenerate parabola, $x^2 = 0$, which is the y - axis taken twice.

If $\beta \gtrless ic$, the equation represents a system of imaginary confocal parabolas having the common focus F_1 (o, ic).

D. To find the system of parabolas which is tangent to F_1I_1 and F_2I_2 and has $x = 0$ as the common axis.

Take $a \equiv F_1I_1 \equiv y - ix + ic = 0.$ (15.)

$b \equiv F_2I_2 \equiv y + ix + ic = 0.$ (16.)

Comparing (15.) with (10.) and (16.) with (11.), we see $m = i$ and $k = -ic$. Substituting these values in (12.),

V. $$x^2 = 4(-ic - \beta)(y - \beta).$$

If $\beta = -ic$,

The equation represents the real degenerate parabola, $x^2 = 0$, which is the y - axis taken twice.

If $\beta \gtrless -ic$,

The equation represents a system of imaginary confocal parabolas, having the common focus F_2' (o, $-ic$).

Therefore we may state the following general results:--

(a) A system of ellipses and hyperbolas, which is tangent to the four lines, which join two real points to the circular points I_1 and I_2, has those two real points as common foci, the other two foci, which are imaginary, being the other two intersections of these imaginary lines.

(b) A system of parabolas, which is tangent to the two lines which join a real point, F, with the two circular points I_1 and I_2 and has a common axis through F, has F as a common focus. Also if the point F is imaginary, the system of parabolas consists of imaginary confocal parabolas.

(c) Conversely, the lines, which join the foci of a conic to the circular points I_1 and I_2, are tangent to the conic.

5. An Application of the Isotropic Lines drawn through the focus of a conic and its relation to a more general application made by Laguerre.

Proof of the Theorem:-

If two fixed tangents to a conic are cut by any other tangent to the conic, the distance along the variable tangent between its intersections with the two fixed tangents always appears under the same angle from a focus of the conic.

Given the two fixed tangents l_1 and l_2 and the focus F of the conic K. Fig. (13).

Draw any other tangent to K, which cuts l_1 and l_2 in A_1 and A_2 respectively. Take this tangent as the x-axis. Draw the y-axis through the focus F and let the ordinate of $F = f$. Also let the abscissas of A_1 and A_2 be a_1 and a_2 respectively. Since the x-axis is any tangent to K, f, a_1, and a_2 will vary with the different possible positions of the axes.

Through F, draw the two isotropic lines i_1 and i_2, which pass through the circular points and intersect the x-axis in B_1 and B_2 respectively. These lines are also tangent to K. Denote the abscissas of B_1 and B_2 by b_1 and b_2, respectively. Since the circular points are always fixed with respect to any axes, as the x-axis takes different positions, the lines i_1 and i_2 always keep their constant slopes $+i$ and $-i$ and $b_1 = -b_2$.

In order to prove that the anharmonic ratio of the points A_1, A_2, B_1, B_2 is constant, we make use of the following theorem:--

If two fixed tangents S and T be intersected by a line P which envelopes a curve of the second class tangent to S and T, the ranges (SP) and (TP) on S and T are projective. Therefore four

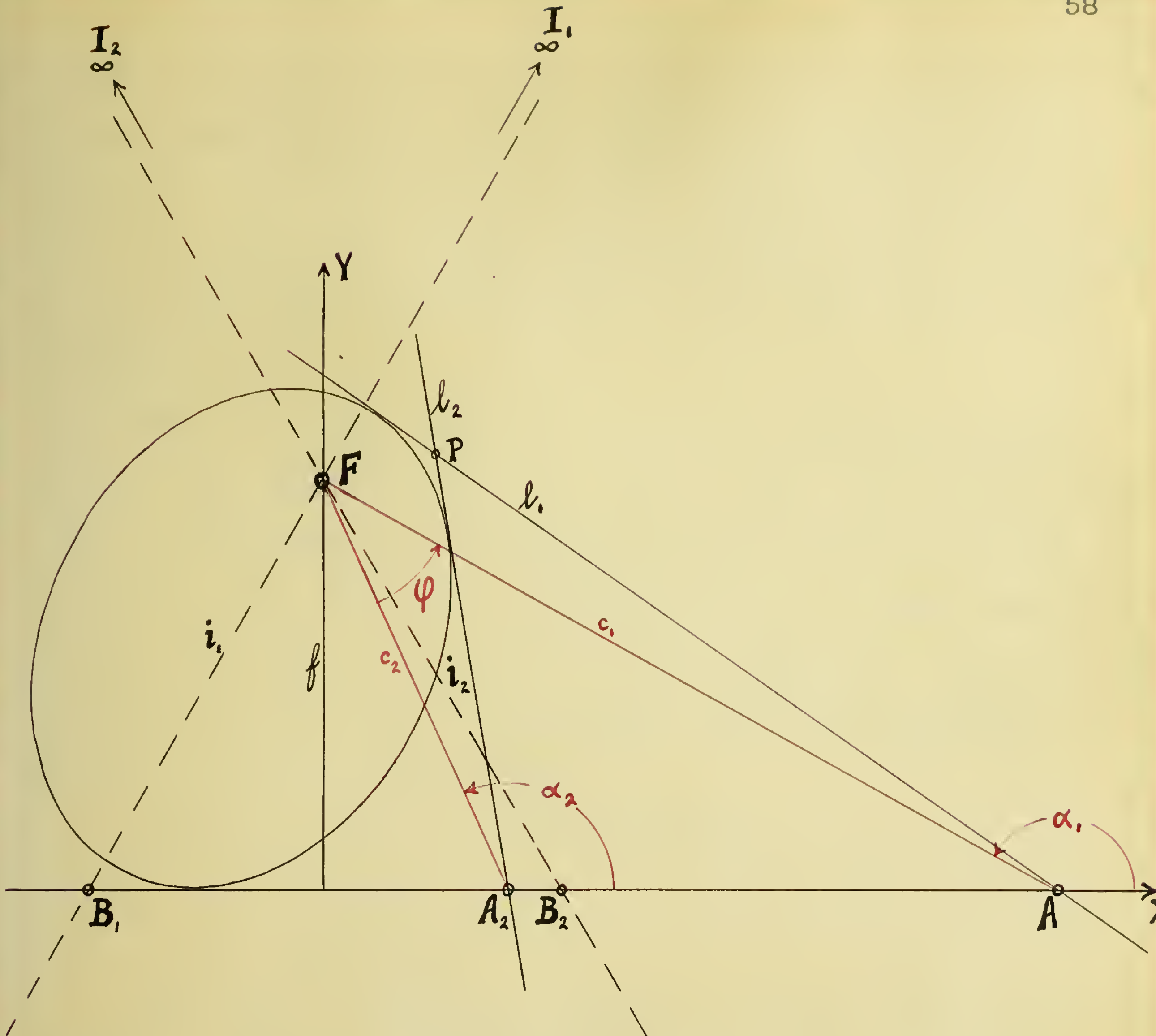
I_2
∞
I_1
∞
Y
l_2
P
F
l_1
φ
c_1
i_1
f
c_2
i_2
α_2
α_1
B_1
A_2
B_2
A
X

Fig. 13.

points of the range (S P) have the same anharmonic ratio as the corresponding four points of the range (T P). Also if the four tangents P, which form the above point ranges are intersected by any other tangent to the given conic, the four points of intersection have the same anharmonic ratio as the corresponding ranges on S and T.

Given the four tangents to the conic K intersecting the x-axis in the points A_1, A_2, B_1, and B_2, which form a point range with the anharmonic ratio $(A_1A_2B_1B_2)$. From the above theorem, as the x-axis takes different positions, the point ranges formed by the intersections of the x-axis with the four given tangents have the same anharmonic ratio $(A_1A_2B_1B_2)$. Therefore $(A_1A_2B_1B_2) = k$ for all positions of the x-axis.

From the figure we get the equations

$$i_1 \equiv y - f = ix,$$

$$i_2 \equiv y - f = - ix.$$

From these we get the values of b_1 and b_2, which are the intercepts of i_1 and i_2 respectively on the x-axis, as follows:-

When $y = 0$, $\quad b_1 \equiv x = if$

$$b_2 \equiv x = -if.$$

Now take the relation $(A_1A_2B_1B_2) = k$.

$$\frac{A_1B_1}{A_2B_1} \Big/ \frac{A_1B_2}{A_2B_2} = k,$$

$$\frac{b_1 - a_1}{b_1 - a_2} \Big/ \frac{b_2 - a_1}{b_2 - a_2} = k,$$

$$\frac{if - a_1}{if - a_2} \Big/ \frac{- if - a_1}{- if - a_2} = k,$$

$$\frac{a_1 - if}{a_2 - if} \Big/ \frac{a_1 + if}{a_2 + if} = k,$$

$$\frac{a_1 - if}{a_2 - if} = k\,\frac{a_1 + if}{a_2 + if}.$$

$$a_1a_2 - ifa_2 + ifa_1 + f^2 = ka_1a_2 + kifa_2 - kifa_1 + kf^2,$$

$$(a_1 - if - ka_1 - kif)\,a_2 = kf^2 - f^2 - kifa_1 - ifa_1.$$

$$\therefore \quad a_2 = \frac{(k-1)\,f^2 - (k+1)\,ifa_1}{(1-k)a_1 - (k+1)\,if}.$$

From Fig. (13.)

$$\varphi = \alpha_1 - \alpha_2.$$

$$\tan\varphi = \frac{\tan\alpha_1 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2}.$$

$$\tan\alpha_1 = -\frac{f}{a_1},$$

$$\tan\alpha_2 = -\frac{f}{a_2},$$

$$= -\frac{f}{\dfrac{(k-1)f^2 - (k+1)ifa_1}{(1-k)a_1 - (k+1)if}},$$

$$= -\frac{(1-k)a_1 - (k+1)if}{(k-1)f - (k+1)ia_1}.$$

$$\therefore \tan\varphi = \frac{-\dfrac{f}{a_1} + \dfrac{(1-k)a_1 - (k+1)if}{(k-1)f - (k+1)ia_1}}{1 + \dfrac{f\left[(1-k)a_1 - (k+1)if\right]}{a_1\left[(k-1)f - (k+1)ia_1\right]}}.$$

$$\tan\varphi = \frac{-(k-1)f^2 + (k+1)ia_1f + (1-k)a_1^2 - (k+1)ia_1f}{(k-1)a_1f - (k+1)ia_1^2 + (1-k)a_1f - (k+1)if^2},$$

$$= \frac{-(k-1)(a_1^2 + f^2)}{-i(k+1)(a_1^2 + f^2)} = \frac{i(1-k)}{k+1}.$$

Therefore $\varphi = \tan^{-1}\left[\dfrac{i(1-k)}{(k+1)}\right]$, which remains constant for any two fixed tangents to the conic K, which was to be proved.

This theorem may be generalized to Laguerre's Theorem, which is as follows:-

An angle formed by any two lines, real or imaginary, is equal to $\frac{1}{2I}$ $\log_e k$, in which k is the anharmonic ratio formed by the sides of the angle and the isotropic lines joining its vertex to the circular points I_1 and I_2.*

To show this we proceed as follows:-

Given the theorem of Pappus:-

The anharmonic ratio of any four concurrent rays is equal to the anharmonic ratio of four points formed by the intersection of any transversal with these rays. Then designating the sides of the

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*M. E. Laguerre.
Nouvelles Annales de Mathematiques, 1853.
Oeuvres, Vol. II. p. 9, published in Paris, 1905.

angle by c_1 and c_2 in Fig. (13.), we know

$$(c_1 c_2 i_1 i_2) = (A_1 A_2 B_1 B_2) = k.$$

In Fig. (14.), let θ be any angle, such that $(c'_1 c'_2 i_1 i_2) = k$ $= (c_1 c_2 i_1 i_2)$.

To show that $\theta = \varphi$.

Given the relations,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

$$\therefore \quad \tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})} = \frac{e^{2i\theta} - 1}{i(e^{2i\theta} + 1)}.$$

$$\therefore \quad \theta = \tan^{-1}\left[\frac{e^{2i\theta} - 1}{i(2^{i\theta} + 1)}\right]. \qquad (1.)$$

From Laguerre's Theorem,

$$\theta = \frac{1}{2i}\log k.$$

$$\log k = 2i\theta.$$

$\therefore \quad k = e^{2i\theta}$, Substituting this value in (1.),

$$\theta = \tan^{-1}\left[\frac{k-1}{i(k+1)}\right],$$

$$= \tan^{-1}\left[\frac{i(1-k)}{(k+1)}\right].$$

Comparing this value with that of φ, we see

$$\theta = \varphi,$$

which establishes the fact which we were to prove.

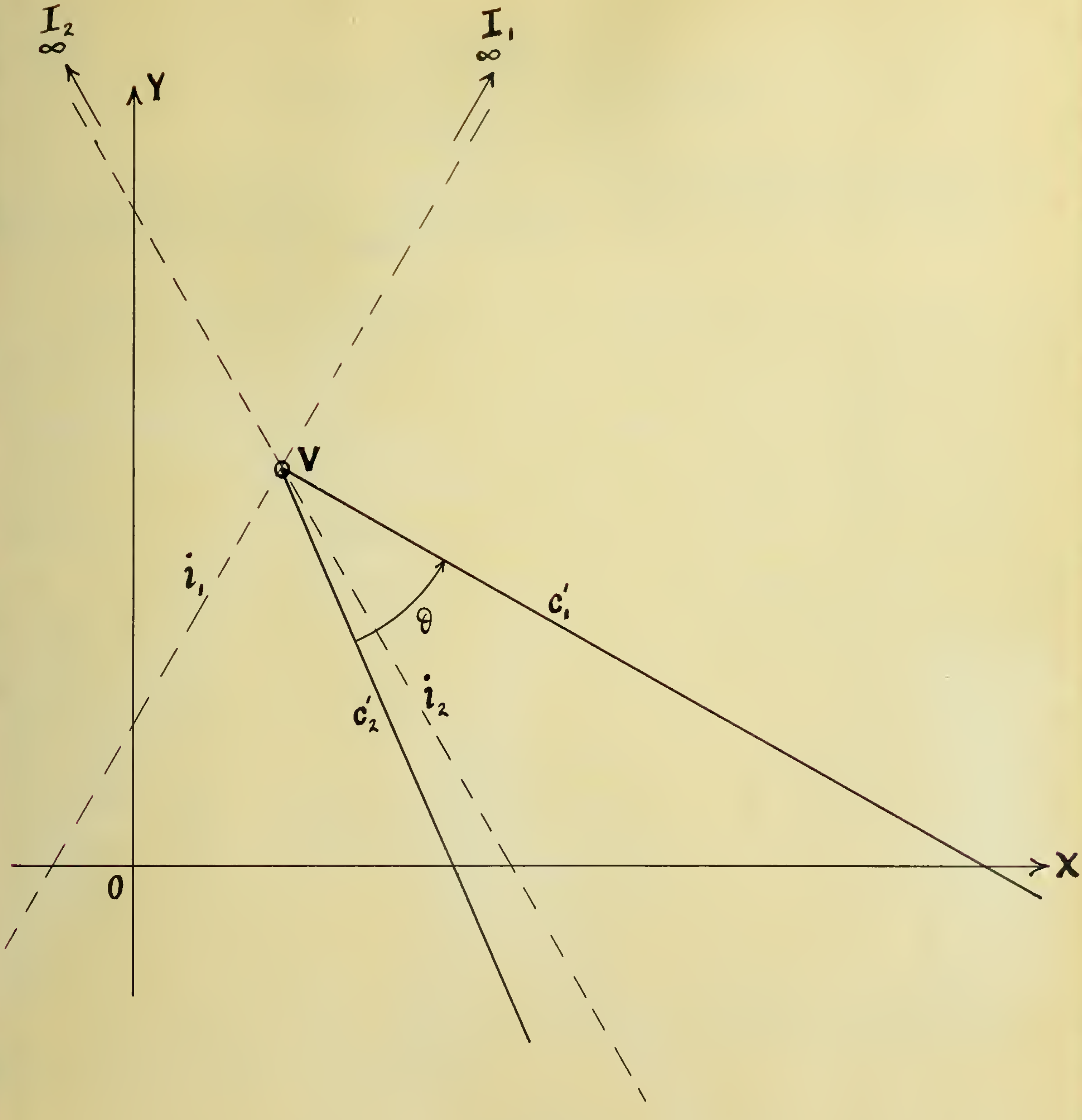

Fig. 14.

CHAPTER I I I .

A DISCUSSION OF THE EQUATIONS OF ANALYTIC CONICS.

1. To distinguish these analytically generalized conics, in which the variables are both complex, from the ordinary conics, they shall be called <u>Analytic</u> <u>Conics</u>.

For the sake of simplicity, the equations of the conics, which have their centers at the origin and their axes coinciding with the co-ordinate axes, may be used without loss of generality, for by a collineation they may be transformed to any other position in the plane without changing the relative position of the figures.

I shall discuss the mapping of the two complex planes, w and z, by the usual method used in the theory of functions,* according to the following relations:

$$w^2 + z^2 - \alpha^2 = 0 , \qquad \text{The Circle,}$$

$$\alpha^2 w^2 + \beta^2 z^2 - \alpha^2\beta^2 = 0, \qquad \text{The Ellipse,}$$

$$\alpha^2 w^2 - \beta^2 z^2 - \alpha^2\beta^2 = 0, \qquad \text{The Hyperbola,}$$

$$w^2 - 2\alpha z = 0, \qquad \text{The Parabola,}$$

in which $w = u + iv$, $z = x + iy$, $\alpha = a + ib$, and $\beta = c + id$.

Since each of the equations contains w in the second degree only, we can write, in general,

$$w = \pm\sqrt{A + iB} ,$$

in which A and B are different polynomials in x and y, obtained respectively from each of the above relations.

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*E. J. Townsend - "Lectures and Functions of a Complex Variable."
Harkness and Morley - "Introduction to the Theory of Analytic Functions."

But since $w = u + iv$,

$$u + iv = \pm\sqrt{A + iB}.$$

$$u^2 + 2iuv - v^2 = A + iB.$$

Equating the real parts,

$$u^2 - v^2 = A \qquad (a)$$

and the imaginary parts,

$$2uv = B. \qquad (b)$$

Solving these equations for u and v,

From (b), $$v = \frac{B}{2u}$$

$$v^2 = \frac{B^2}{4u^2}.$$

Substituting in (a),

$$u^2 - \frac{B^2}{4u^2} = A.$$

$$4u^4 - 4Au^2 - B^2 = 0$$

$$u^2 = \frac{4A \pm 4\sqrt{A^2 + B^2}}{8}$$

$$= \frac{A \pm \sqrt{A^2 + B^2}}{2}$$

We may also express $w = \pm\sqrt{\rho e^{i\theta}}$, in which $\rho = \sqrt{A^2 + B^2}$ and $\theta = \tan\frac{B}{A}$.

Since ρ is always positive, $\sqrt{A^2 + B^2}$ is always positive.

Therefore $$u^2 = \frac{A + \sqrt{A^2 + B^2}}{2}$$

$$u = \pm\sqrt{\frac{A + \sqrt{A^2 + B^2}}{2}}$$

Since $$v = \frac{B}{2u},$$

$$v = \pm \frac{B}{\sqrt{2(a + \sqrt{A^2 + B^2})}}.$$

We will, in general, consider two cases. Case I, when v equals a constant, and Case II, when u equals a constant.

Case I. Let $v = k$.

Then $$k = \pm \frac{B}{\sqrt{2(A + \sqrt{A^2 + B^2})}}.$$

$$2k^2\left[A + \sqrt{A^2 + B^2}\right] = B^2$$

$$2k^2 \sqrt{A^2 + B^2} = B^2 - 2k^2A$$

$$4k^4A^2 + 4k^4B^2 = B^4 - 4k^2AB^2 + 4k^4A^2$$

$$B^4 - 4k^4B^2 - 4k^2AB^2 = 0$$

$$B^2 - 4k^4 - 4k^2A = 0.$$

Case II. Let $u = k$.

Then $$k = \pm \sqrt{\frac{A + \sqrt{A^2 + B^2}}{2}}.$$

$$2k^2 = A + \sqrt{A^2 + B^2}$$

$$\sqrt{A^2 + B^2} = 2k^2 - A$$

$$A^2 + B^2 = 4k^4 - 4k^2A + A^2$$

$$B^2 - 4k^4 + 4k^2A = 0.$$

2. The Circle.

Given the equation $w^2 + Z^2 = \alpha^2$.

From which $$w = \pm \sqrt{\alpha^2 - z^2}$$

$$= \pm \sqrt{a^2 + 2iab - b^2 - x^2 - 2ixy + y^2}.$$

But in general $w = \pm\sqrt{A + iB}$.

Therefore $A + iB = a^2 + 2iab - b^2 - x^2 - 2ixy + y^2$.

Equating the real parts,

$$A = a^2 - b^2 - x^2 + y^2 ; \qquad (1)$$

and the imaginary parts,

$$B = 2ab - 2xy. \qquad (2)$$

A general discussion of the function

$$z^2 = \alpha^2 - w^2.*$$

A complete discussion of the two to two correspondence given by $z^2 = \alpha^2 - w^2$ would lead us to consider two Riemann Surfaces:- a two sheeted surface spread over the z-plane and a two sheeted surface spread over the w-plane. There would then be a one to one correspondence between the places (w, z) of the w-surface and the places (z, w) of the z-surface.

I will, however, only give a brief discussion of the branch points of the plane, considering in particular the behavior of the lines parallel to the axes passing through these critical points.

From the above function, we see that the two values of z are equal when $w = \pm\alpha$ and ∞.

Upon investigation, we see that $w = \pm\alpha$ are branch points, for when w describes a circuit around either one of them separately there is a change of z from one value to the other. However, this is not the case with $w = \infty$, for when w describes a

*Harkness and Morley. "Introduction to Analytic Functions." Example III. Page 282.

circuit around both $+\alpha$ and $-\alpha$, z returns to the same value from which it started. The point $w = \infty$, however, has been defined as the nodal case. The sheets of the w-surface touch at $w = \infty$ and so do those of the z-surface at $z = \infty$, but there is no other connection between the sheets in the neighborhoods of these points.

Consider the mapping of the w-plane upon the z-plane by means of the function $w^2 = \alpha^2 - z^2$.

First, let w describe the line $v = k$, where k is any constant. From I, (1), and (2),

$$4(ab - xy)^2 - 4k^4 - 4k^2(a^2 - b^2 - x^2 + y^2) = 0,$$

$$a^2b^2 - 2abxy + x^2y^2 - k^4 - a^2k^2 + b^2k^2 + k^2x^2 - k^2y^2 = 0,$$

$$x^2y^2 + k^2x^2 - 2abxy - k^2y^2 + a^2b^2 - k^4 - a^2k^2 + b^2k^2 = 0.$$

Solving first for y and then for x,

$$y = \frac{abx \pm \sqrt{a^2b^2x^2 - (x^2 - k^2)(a^2b^2 - k^4 - a^2k^2 + b^2k^2 + k^2x^2)}}{(x^2 - k^2)},$$

and

$$x = \frac{aby \pm \sqrt{a^2b^2y^2 - (y^2 + k^2)(a^2b^2 - k^4 - a^2k^2 + b^2k^2 - k^2y^2)}}{(y^2 + k^2)}.$$

Therefore, by this transformation the straight lines parallel to the u-axis, v_k in Fig. (15), in the w-plane, are transformed into the quartics, v'_k, in the z-plane.

The, let w describe the straight line $u = k$.
From II, (1), and (2),

w-plane.

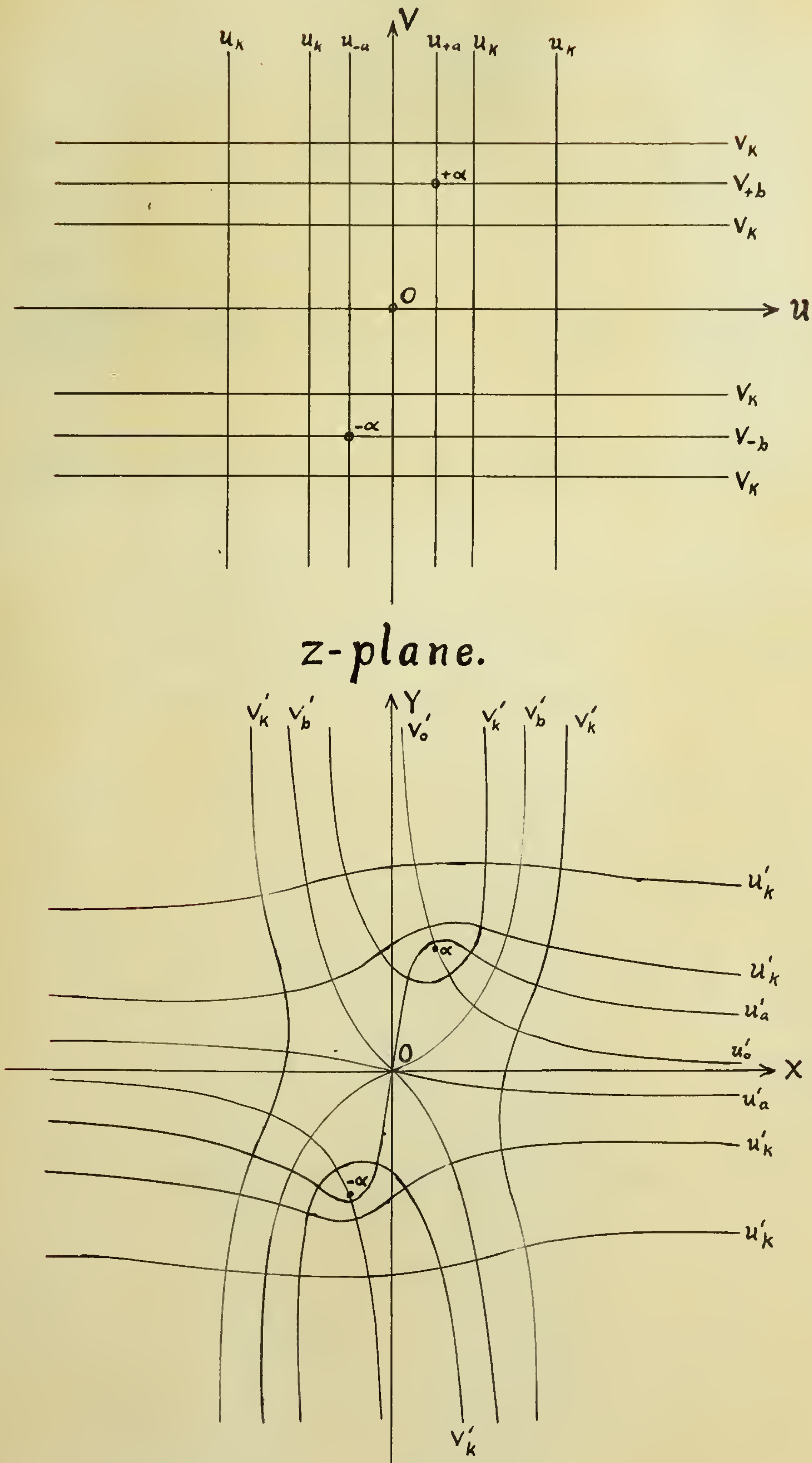


Fig. 15.

$$4(ab - xy)^2 - 4k^4 + 4k^2(a^2 - b^2 - x^2 + y^2) = 0.$$

$$a^2b^2 - 2abxy + x^2y^2 - k^4 + a^2k^2 - b^2k^2 - k^2x^2 + k^2y^2 = 0$$

$$x^2y^2 - k^2x^2 - 2abxy + k^2y^2 + a^2b^2 - k^4 + a^2k^2 - b^2k^2 = 0.$$

Solving first for y and then for x, we get

$$y = \frac{abx \pm \sqrt{a^2b^2x^2 - (x^2 + k^2)(a^2b^2 - k^4 + a^2k^2 - b^2k^2 - k^2x^2)}}{(x^2 + k^2)},$$

and

$$x = \frac{aby \pm \sqrt{a^2b^2y^2-(y^2 - k^2)(a^2b^2 - k^4 + a^2k^2 - b^2k^2 + k^2y^2)}.}{(y^2 - k^2)}$$

From this, it is seen that by this transformation the straight lines parallel to the v-axis in the w-plane, u_k, are transformed into the quarters, u'_k, in the z-plane (Fig.15).

We know from the properties of conformal mapping that, in general, angles are preserved by a transformation of this kind from one plane to the other. Therefore, we may say, in general, that the quartics, u'_k, intersect the quartics, v'_k, at right angles, since the lines, u_k, are orthogonal to the lines, v_k. However, we will find some particular cases when this isogonality breaks down. To show this, consider the lines in the w-plane parallel to the axes which pass through the branch points $+\alpha$ and $-\alpha$.

We have, $w = \pm \alpha$.

$$u + iv = \pm (a + ib).$$

Therefore $\quad u = \pm a,$

$$v = \pm b.$$

Now, let w describe either of the straight lines $v = \pm b$.

From I, (1), and (2), we get

$$4(ab - xy)^2 - 4b^4 - 4b^2(a^2 - b^2 - x^2 + y^2) = 0,$$

$$a^2b^2 - 2abxy + x^2y^2 - b^4 - a^2b^2 + b^4 + b^2x^2 - b^2y^2 = 0,$$

$$x^2y^2 + b^2x^2 - b^2y^2 - 2abxy = 0.$$

Therefore, by this transformation the straight lines v_{-b} and v_{+b}, which pass through the points $-\alpha$ and $+\alpha$, respectively, are each transformed into the quartic represented in Fig. 15 by v'_b, which consists of two branches which unite at the point $z = 0$.

Second let w describe either of the straight lines $u = \pm a$

From II, (1), and (2),

$$4(ab - xy)^2 - 4a^4 + 4a^2(a^2 - b^2 - x^2 + y^2) = 0,$$

$$a^2b^2 - 2abxy + x^2y^2 - a^4 + a^4 - a^2b^2 - a^2x^2 + a^2y^2 = 0,$$

$$x^2y^2 - a^2x^2 + a^2y^2 - 2abxy = 0.$$

Therefore, by this transformation, the straight lines u_{-a} and u_{+a}, passing through the branch points $-\alpha$ and $+\alpha$ respectively, are each transformed into the quartic, represented in Fig. 15 by u'_a, consisting of two branches which also unite at the point $z = 0$.

From the development of rhizic curves by William Walton,* we know that the two branches of v'_b are perpendicular to each other at $z = 0$, likewise, the branches of u'_a are perpendicular to each other at the same point.

Also the tangents to v'_b at $z = 0$ bisect the angles formed by the tangents to u'_a at $z = 0$ and conversely. Therefore at these

*The Quarterly Journal of Pure and Applied Mathematics, Vol. XI. Page 93.

two branch points we see that the isogonality breaks down.

We may also expect another irregularity in the conformal mapping of the w-plane upon the z-plane at the point $w = 0$, for this is the point into which the branch points of the z-plane are transformed in the converse mapping.

We will investigate this point by mapping the two lines which pass through it, $u = 0$ and $v = 0$.

First, we will let w describe the line $v = 0$.

From I and (2),

we get $xy = ab$ as the equation of the corresponding curve in the z-plane, which is an equilateral hyperbola passing through the points $+\alpha$ and $-\alpha$.

Then, let w describe the line $u = 0$.

From II, and (2),

we again get $xy = ab$ for the equation of the corresponding curve in the z-plane.

But we must distinguish between the part of this hyperbola into which the line $v = 0$ maps and that into which $u = 0$ maps.

In order to do this we will start with the equations which we developed in the general discussion of the functions under consideration:-

$$u = \pm\sqrt{\frac{A + \sqrt{A^2 + B^2}}{2}}.$$

$$v = \pm\frac{B}{\sqrt{2(A + \sqrt{A^2 + B^2})}}.$$

Also

$$A = a^2 - b^2 - x^2 + y^2.$$

$$B = 2ab - 2xy.$$

Therefore, $u = \pm \sqrt{\dfrac{a^2-b^2-x^2+y^2+\sqrt{(a^2-b^2-x^2+y^2)^2+4(ab-xy)^2}}{2}}$,

and $v = \pm \dfrac{2ab - 2xy}{\sqrt{2\left[a^2 - b^2 - x^2 + y^2 + \sqrt{(a^2 - b^2 - x^2 + y^2)^2 + 4(ab - xy)^2}\right]}}$

From the symmetry of the equation, $w^2 + z^2 = \alpha^2$, we can write,

$$x = \pm \sqrt{\frac{a^2 - b^2 - u^2 + v^2 + \sqrt{(a^2 - b^2 - u^2 + v^2)^2 + 4(ab - uv)^2}}{2}},$$

and

$$y = \pm \frac{2ab - 2uv}{\sqrt{2\left[(a^2 - b^2 - u^2 + v^2) + \sqrt{(a^2 - b^2 - u^2 + v^2)^2 + 4(ab - uv)^2}\right]}}.$$

First, let $v = 0$,

Then

$$x = \pm \sqrt{\frac{a^2 - b^2 - u^2 + \sqrt{(a^2 - b^2 - u^2)^2 + 4a^2b^2}}{2}},$$

and

$$y = \pm \frac{2ab}{\sqrt{2\left[(a^2 - b^2 - u^2) + \sqrt{(a^2 - b^2 - u^2)^2 + 4a^2b^2}\right]}}.$$

By letting u approach the value $u = 0$ from positive infinity along the line $v = 0$, we see that the point z in the z-plane starts at $\pm i\infty$ and traces the equilateral hyperbola $xy = ab$, until, when $u = 0$, z becomes $\pm (a + ib)$ respectively.

Then, we will let u approach the value $u = 0$ from negative infinity along the line $v = 0$ and we see that the point z again starts at $\pm i\infty$ and traces the same equilateral hyperbola $xy = ab$ until, when $u = 0$, z again becomes $\pm (a + ib)$; thus in both cases, describing the same portion of the equilateral hyperbola.

Therefore, we see that the line $v = 0$, passing through the point $w = 0$, maps into that portion of the equilateral hyperbola,

$xy = ab$, between the points $+\alpha$ and $+i\infty$ and $-\alpha$ and $-i\infty$, tracing each curve twice. Fig. (15).

Second, let $u = 0$

Then the equations of x and y become

$$x = \pm\sqrt{\frac{a^2 - b^2 + v^2 + \sqrt{(a^2 - b^2 + v^2)^2 + 4a^2b^2}}{2}},$$

and

$$y = \pm\frac{2ab}{\sqrt{2\left[(a^2 - b^2 + v^2) + \sqrt{(a^2 - b^2 + v^2)^2 + 4a^2b^2}\right]}}.$$

By letting v approach the value $v = 0$ from positive infinity along the line $u = 0$, we see that the point z, in the z-plane, starts at $\pm\infty$ and traces the equilateral hyperbola, $xy = ab$, until, when $v = 0$, z becomes $\pm(a + ib)$ respectively.

Then let v approach the value $v = 0$ from negative infinity along the line $u = 0$, and we see that z again starts at $\pm\infty$ and traces the same equilateral hyperbola, $xy = ab$, until, when $v = 0$, z again becomes $\pm(a + ib)$; thus in both cases tracing the same portion of the equilateral hyperbola.

Therefore, we see that the line $u = 0$, passing through the point $w = 0$, maps into that portion of the equilateral hyperbola, $xy = ab$, between the points $+\alpha$ and $+\infty$; also between $-\alpha$ and $-\infty$, tracing each curve twice. Fig. (15).

Therefore, at the point $w = 0$, which maps into the two points $+\alpha$ and $-\alpha$ in the z-plane, we have the isogonality breaking down in such a way that the orthogonal lines passing through $w = 0$, map into curves in the z-plane which join each other at $\pm\alpha$ such that both have the same slope at these points, thus forming a continuous

hyperbola.

Since $w^2 + z^2 = \alpha^2$ is a symmetrical function with respect to z and w, and there is a two to two correspondence between the w and z planes, we say that the two planes are in a quadratic involutoric relation and therefore the converse mapping is exactly the same as that discussed.

3. The Ellipse.

Taking the equation of the ellipse

$$\alpha^2 w^2 + \beta^2 z^2 - \alpha^2 \beta^2 = 0,$$

in which $\alpha = a + ib$, and $\beta = c + id$,

we get $\quad w = \pm\sqrt{\dfrac{\alpha^2\beta^2 - \beta^2 z^2}{\alpha^2}}.$

$$w = \pm\sqrt{\frac{(a^2 + 2iab - b^2)(c^2 + 2icd - d^2) - (c^2 + 2icd - d^2)(x^2 + 2ixy - y^2)}{a^2 + 2iab - b^2}}$$

$$= \pm\sqrt{\frac{a^2c^2 + 2ia^2cd - a^2d^2 - 2iabc^2 - 4abcd - 2iabd^2 - b^2c^2 - 2ib^2cd}{a^2 + 2iab - b^2}}$$

$$\frac{+b^2d^2 - c^2x^2 - 2ic^2xy + c^2y^2 - 2icdx^2 + 4cdxy + 2icdy^2 + d^2x^2}{a^2 + 2iab - b^2}$$

$$\frac{+2id^2xy - d^2y^2}{a^2 + 2iab - b^2}$$

$$= \pm\sqrt{\frac{(a^2 - b^2)(a^2c^2 - a^2d^2 - 4abcd - b^2c^2 + b^2d^2 - c^2x^2 + c^2y^2}{(a^2 + b^2)^2}}$$

$$\frac{+ 4cdxy + d^2x^2 - d^2y^2) + 4ab(a^2cd + abc^2 - abd^2 - b^2cd - c^2xy}{(a^2 + b^2)^2}$$

$$\frac{- cdx^2 + cdy^2 + d^2xy) + i2(a^2 - b^2)(a^2cd + abc^2 - abd^2 - b^2cd}{(a^2 + b^2)^2}$$

$$\frac{-c^2xy - cdx^2 + cdy^2 + d^2xy) - i2ab(a^2c^2 - a^2d^2 - 4abcd - b^2c^2 + b^2d^2}{(a^2 + b^2)^2}$$

$$\frac{-c^2x^2 + c^2y^2 + 4\,cdxy + d^2x^2 - d^2y^2)}{(a^2 + b^2)^2}$$

Since in general $w = \pm\sqrt{A + iB}$, we get

$$A = \frac{(a^2 - b^2)(a^2c^2 - a^2d^2 - 4abcd - b^2c^2 + b^2d^2 - c^2x^2 + c^2y^2}{(a^2 + b^2)^2}$$

$$\frac{+ 4cdxy + d^2x^2 - d^2y^2) + 4ab(a^2cd + abc^2 - abd^2 - b^2cd - c^2xy}{(a^2 + b^2)^2}$$

$$\frac{-cdx^2 + cdy^2 + d^2xy)}{(a^2 + b^2)^2},$$

which reduces to

$$A = \left[\frac{(a^2 - b^2)(d^2 - c^2) - 4abcd}{(a^2 + b^2)^2}\right]x^2 - \left[\frac{4ab(c^2 - d^2) - 4cd(a^2 - b^2)}{(a^2 + b^2)^2}\right]xy$$

$$- \left[\frac{(a^2 - b^2)(d^2 - c^2) - 4abcd}{(a^2 + b^2)^2}\right]y^2 - (d^2 - c^2);$$

and

$$B = \frac{2(a^2 - b^2)(a^2cd + abc^2 - abd^2 - b^2cd - c^2xy - cdx^2 + cdy^2 + d^2xy)}{(a^2 + b^2)^2}$$

$$- \frac{2ab(a^2c^2 - a^2d^2 - 4abcd - b^2c^2 + b^2d^2 - c^2x^2 + c^2y^2 + 4cdxy}{(a^2 + b^2)^2}$$

$$+ \frac{d^2x^2 - d^2y^2)}{(a^2 + b^2)^2},$$

which reduces to

$$B = 2\left[\frac{ab(c^2 - d^2) - cd(a^2 - b^2)}{(a^2 + b^2)^2}\right]x^2 + 2\left[\frac{(a^2 - b^2)(d^2 - c^2) - 4abcd}{(a^2 + b^2)^2}\right]xy$$

$$-2\left[\frac{ab(c^2 - d^2) - cd(a^2 - b^2)}{(a^2 + b^2)^2}\right]y^2 + 2\,cd.$$

These may be written

$A = lx^2 - 4mxy - ly^2 - n,$ (3.)

$B = 2mx^2 + 2lxy - 2my^2 - 2p,$ (4.)

in which

$$l = \frac{(a^2 - b^2)(d^2 - c^2) - 4abcd}{(a^2 + b^2)^2}$$

$$m = \frac{ab(c^2 - d^2) - cd(a^2 - b^2)}{(a^2 + b^2)^2}$$

$$n = d^2 - c^2$$

$$p = -cd$$

In discussing the function

$$z^2 = \frac{\alpha^2}{\beta^2}(\beta^2 - w^2),$$

we notice that it has the same general form as $z^2 = \alpha^2 - w^2$. Therefore the general discussion given in regard to the branch points of the latter function also applies to this function.

Therefore, we see at once that the branch points in the w-plane are $w = \pm\beta$ and the nodal case is again $w = \infty$.

I will discuss the mapping of the w-plane upon the z-plane by means of the function $w^2 = \frac{\beta^2}{\alpha^2}(\alpha^2 - z^2)$ as in the preceding case, by giving constant values to u and v and finding the corresponding curve in the z-plane.

First, let w describe the straight line $v = k$.

From I, (3), and (4),

$$4(mx + lxy - my^2 - p)^2 - 4k^4 - 4k^2(lx^2 - 4mxy - ly^2 - n) = 0$$

$$m^2x^4 + 2lmx^3y + (l^2 - 2m^2)x^2y^2 - 2lmxy^3 + m^2y^4 - 2mpx^2 - 2lpxy$$
$$+ 2mpy^2 + p^2 - k^4 - k^2lx^2 + 4mk^2xy + k^2ly^2 + k^2n = 0,$$

which may be written

$$m^2x^4 + 2lmx^3y + (l^2 - 2m^2)x^2y^2 - 2lmxy^3 + m^2y^4 - (2mp + k^2l)x^2$$
$$+(4mk^2 - 2lp)xy + (2mp + k^2l)y^2 + p^2 - k^4 + k^2n = 0.$$

Therefore by this transformation the straight lines parallel to the u-axis, v_k, in the w-plane are transformed into the quartics, v'_k, in the z-plane, each of which always has two branches. Fig. (16).

Second, let w describe the straight line $u = k$.

From II, (3), and (4),

$$4(mx^2 + lxy - my^2 - p)^2 - 4k^4 + 4k^2(lx^2 - 4mxy - ly^2 - n) = 0,$$
$$m^2x^4 + 2lmx^3y + (l^2 - 2m^2)x^2y^2 - 2lmxy^3 + m^2y^4 - 2mpx^2 - 2lpxy$$
$$+2mpy^2 + p^2 - k^4 + k^2lx^2 - 4mk^2xy - k^2ly^2 - k^2n = 0$$

or

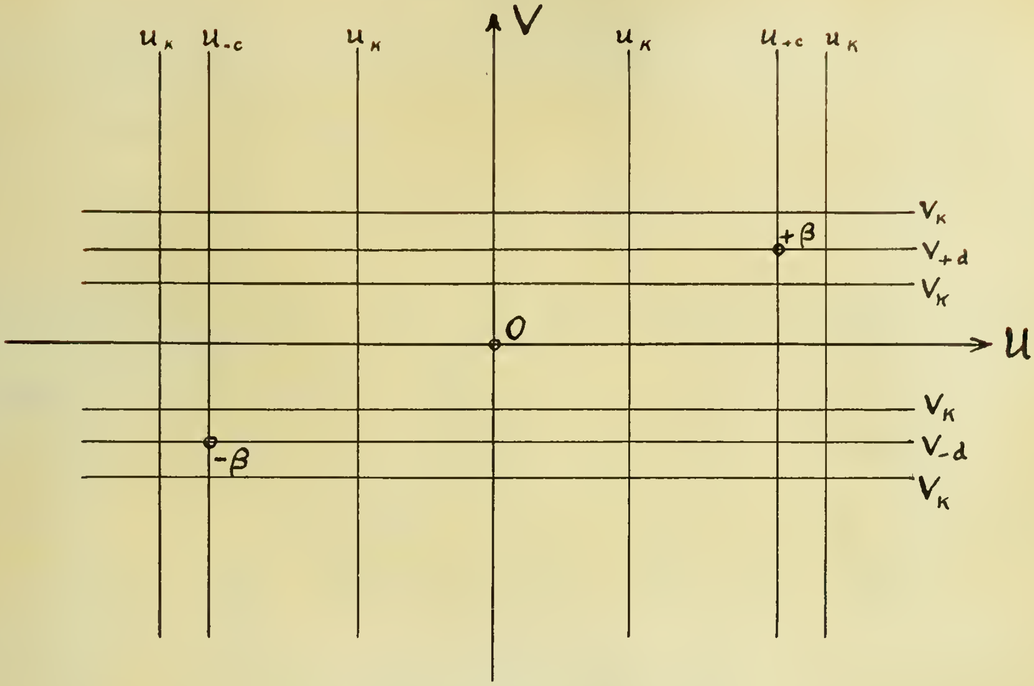
$$m^2x^4 + 2lmx^3y + (l^2 - 2m^2)x^2y^2 - 2lmxy^3 + m^2y^4 - (2mp - k^2l)x^2$$
$$-(4mk^2 + 2lp)xy + (2mp - k^2l)y^2 + p^2 - k^4 - k^2n = 0$$

Therefore, by this transformation, the straight lines parallel to the v-axis, u_k, in the w-plane are transformed into the quartics, u'_k, in the z-plane, each of which always has two branches, Fig. (16).

By this transformation, as in the preceding one, in general the curves u'_k are orthogonal to the curves v'_k, for the lines u_k are orthogonal to v_k in the w-plane. However, in the case of the branch points we again have this isogonality breaking down.

To show this, consider the lines in the w-plane parallel to the axes, which pass through the branch points $w = \pm\beta$.

w-plane.



z-plane.

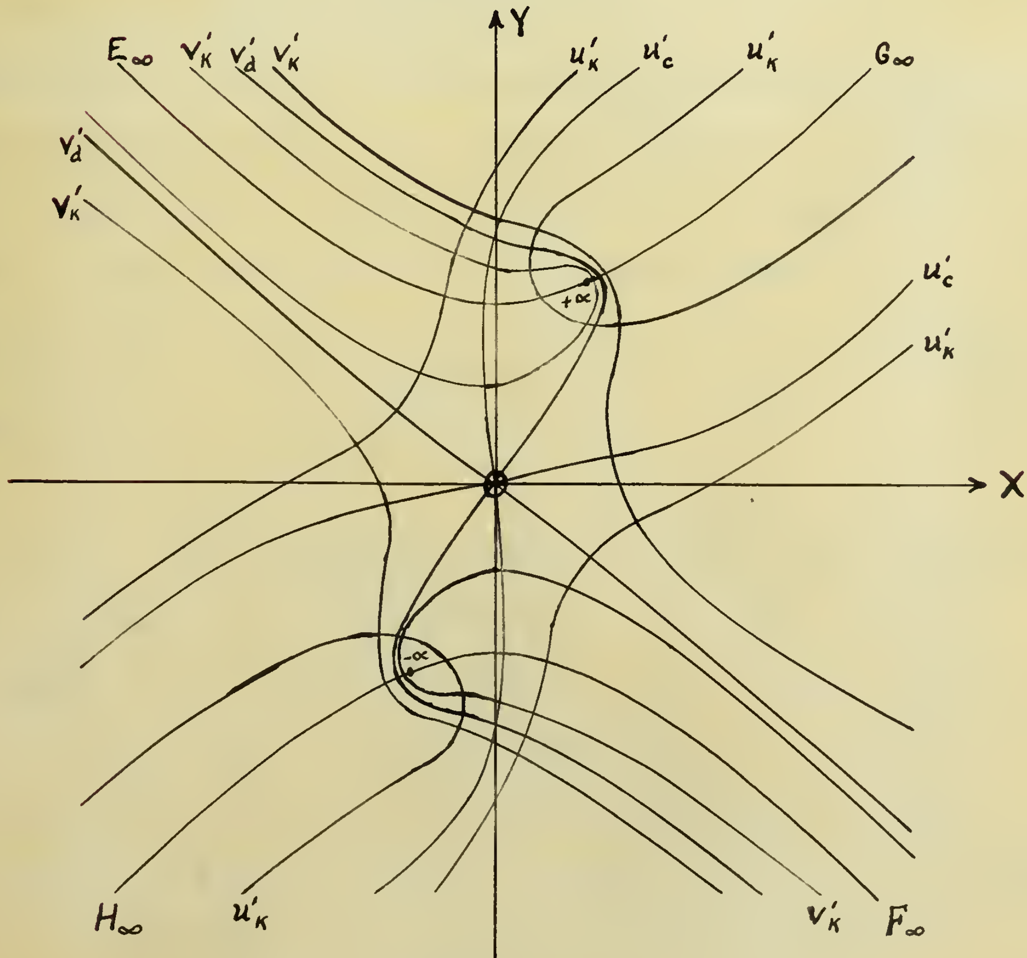


Fig. 16.

We have $w = \pm\beta$.

Then $u+iv = \pm(c + id)$.

$$u = \pm c,$$

$$v = \pm d.$$

Now, let w describe either of the straight lines $v = \pm d$.

From I, $\quad B^2 - 4d^4 - 4d^2A = 0$

From (3), and (4), and remembering $n = d^2 - c^2$ and $p = -cd$,

$$m^2x^4 - 2lmx^3y + (l^2 - 2m^2)x^2y^2 - 2lmxy^3 + m^2y^4 - (2mp + d^2l)x^2$$
$$+(4md^2 - 2lp)xy + (2mp + d^2l)y^2 + c^2d^2 - d^4 + d^4 - c^2d^2 = 0,$$

which reduces to

$$m^2x^4 + 2lmx^3y + (l^2 - 2m^2)x^2y^2 - 2lmxy^3 + m^2y^4 - (2mp + d^2l)x^2$$
$$+(4md^2 - 2lp)xy + (2mp + d^2l)y^2 = 0.$$

Therefore, by this transformation the straight lines v_{-d} and v_{+d}, passing through the branch points $-\beta$ and $+\beta$, respectively, are transformed into the quartic, represented in Fig. (16) by V'_d, consisting of two branches which unite at the point $z = 0$.

Next, let w describe either of the straight lines $u = \pm c$.

From II, $B^2 - 4c^4 + 4c^2A = 0$.

From (3) and (4)

$$m^2x^4 + 2lmx^3y + (l^2 - 2m^2)x^2y^2 - 2lmxy^3 + m^2y^4 - (2mp - c^2l)x^2$$
$$- (4mc^2 + 2lp)xy + (2mp - c^2l)y^2 + c^2d^2 - c^4 - c^2d^2 + c^4 = 0$$

$$m^2x^4 + 2lmx^3y + (l^2 - 2m^2)x^2y^2 - 2lmxy^3 + m^2y^4 - (2mp - c^2l)x^2$$
$$- (4mc^2 + 2lp)xy + (2mp - c^2l)y^2 = 0.$$

Therefore, by this transformation the straight lines u_{-c} and u_{+c} passing through the branch points $-\beta$ and $+\beta$, respectively, are transformed into the quartic, represented in Fig. (16)

by u'_c, consisting of two branches which unite at the point $z = 0$.

In this function, as in the preceding one, the two branches of v'_d are perpendicular to each other at $z = 0$ and also the branches of u'_c are orthogonal at $z = 0$.

Also the tangents to v'_d at $z = 0$ bisect the angles formed by the tangents to u'_c at $z = 0$ and conversely.

We may expect another irregularity in the conformal mapping of this function at the point $w = 0$. If we let w describe the straight line $v = 0$, we see from I and (4) that z describes the equilateral hyperbola

$$mx^2 + lxy - my^2 - p = 0,$$

which passes through the points $+\alpha$ and $-\alpha$. Also, if we let w describe the straight line $u = 0$, we see from II and (4) that z describes the same hyperbola,

$$mx^2 + lxy - my^2 - p = 0.$$

By an investigation similar to that made in regard to the similar case of the circle, we see that the entire hyperbola is not described in each case, but rather that the line $v = 0$ maps into that portion of the hyperbola from $+\alpha$ to E_∞ and $-\alpha$ to F_∞ ; also that the line $u = 0$ maps into that part from $+\alpha$ to G_∞ and $-\alpha$ to H_∞ (Fig.16).

Therefore, at the point $w = 0$, which maps into the two points $+\alpha$ and $-\alpha$ in the z-plane, we have the isogonality breaking down in such a way that the orthogonal lines passing through $w = 0$ map into curves in the z-plane which join each other at $+\alpha$ and $-\alpha$, such that both have the same slope at these points, thus

forming a continuous equilateral hyperbola.

4. The Hyperbola.

Taking the equation of the hyperbola

$$\alpha^2 w^2 - \beta^2 z^2 - \alpha^2 \beta^2 = 0,$$

we get

$$w = \pm\sqrt{\frac{\alpha^2 \beta^2 + \beta^2 z^2}{\alpha^2}},$$

$$w = \pm\sqrt{\frac{(a^2 + 2iab - b^2)(c^2 + 2icd - d^2) + (c^2 + 2icd - d^2)(x^2 + 2ixy - y^2)}{a^2 + 2iab - b^2}}$$

$$= \pm\sqrt{\frac{a^2c^2 + 2ia^2cd - a^2d^2 + 2iabc^2 - 4abcd - 2iabd^2 - b^2c^2 - 2ib^2cd}{a^2 + 2iab - b^2}}$$

$$\frac{+ b^2d^2 + c^2x^2 + 2ic^2xy - c^2y^2 + 2icdx^2 - 4cdxy - 2icdy^2 - d^2x^2}{a^2 + 2iab - b^2}$$

$$\frac{-2id^2xy + d^2y^2}{a^2 + 2iab - b^2}$$

$$= \pm\sqrt{\frac{(a^2 - b^2)(a^2c^2 - a^2d^2 - 4abcd - b^2c^2 + b^2d^2 + c^2x^2 - c^2y^2}{(a^2 + b^2)^2}}$$

$$\frac{-4cdxy - d^2x^2 + d^2y^2) + 4ab(a^2cd + abc^2 - abd^2 - b^2cd + c^2xy}{(a^2 + b^2)^2}$$

$$\frac{+cdx^2 - cdy^2 - d^2xy) + i2(a^2 - b^2)(a^2cd + abc^2 - abd^2 - b^2cd + c^2xy}{(a^2 + b^2)^2}$$

$$\frac{+cdx^2 - cdy^2 - d^2xy) - i2ab(a^2c^2 - a^2d^2 - 4abcd - b^2c^2 + b^2d^2 + c^2x^2}{(a^2 + b^2)^2}$$

$$\frac{-c^2y^2 - 4cdxy - d^2x^2 + d^2y^2)}{(a^2 + b^2)^2}.$$

Since, in general $w = \pm\sqrt{A + iB}$, we get

$$A = \frac{(a^2 - b^2)(a^2c^2 - a^2d^2 - 4abcd - b^2c^2 + b^2d^2 + c^2x^2 - c^2y^2 - 4cdxy}{(a^2 + b^2)^2}$$

$$\frac{- d^2x^2 + d^2y^2) + 4ab(a^2cd + abc^2 - abd^2 - b^2cd + c^2xy + cdx^2 - cdy^2}{(a^2 + b^2)^2}$$

$$\frac{- d^2xy)}{(a^2 + b^2)^2}$$ which reduces to

$$A = \left[\frac{4abcd - (a^2 - b^2)(d^2 - c^2)}{(a^2 + b^2)^2}\right]x^2 + 4\left[\frac{ab(c^2 - d^2) - cd(a^2 - b^2)}{(a^2 + b^2)^2}\right]xy$$

$$- \left[\frac{4abcd - (a^2 - b^2)(d^2 - c^2)}{(a^2 + b^2)^2}\right]y^2 - (d^2 - c^2)$$

and

$$B = \frac{2(a^2 - b^2)(a^2cd + abc^2 - abd^2 - b^2cd + c^2xy + cdx^2 - cdy^2 - d^2xy)}{(a^2 + b^2)^2}$$

$$- \frac{2ab(a^2c^2 - a^2d^2 - 4abcd - b^2c^2 + b^2d^2 + c^2x^2 - c^2y^2 - 4cdxy - d^2x^2 + d^2y^2)}{(a^2 + b^2)^2}$$

which reduces to

$$B = 2\left[\frac{cd(a^2 - b^2) - ab(c^2 - d^2)}{(a^2 + b^2)^2}\right]x^2 - 2\left[\frac{(a^2 - b^2)(d^2 - c^2) - 4abcd}{(a^2 + b^2)^2}\right]xy$$

$$+ 2\left[\frac{ab(c^2 - d^2) - cd(a^2 - b^2)}{(a^2 + b^2)^2}\right]y^2 + 2cd.$$

Then

$A = - lx^2 + 4\,mxy + ly^2 - n,$

$B = - 2\,mx^2 - 2\,lxy + 2\,my^2 - 2\,p.$

Therefore, $A = l\,y^2 + 4\,mxy - l\,x^2 - n,$ (5.)

$B = 2(my^2 - lxy - mx^2 - p).$ (6.)

In discussing the function

$$z^2 = \frac{\alpha^2}{\beta^2}(w^2 - \beta^2),$$

we see from its similarity to the equation of the ellipse, that the branch points in the w-plane are $w = \pm \beta$ and the nodal case is again $w = \infty$.

Then the discussion of the mapping of the w-plane on the z-plane by means of the function $z^2 = \frac{\alpha^2}{\beta^2}(w^2 - \beta^2)$ follows by giving constant values to u and v and finding the corresponding curves in the z-plane.

First, let w describe the straight line $v = k$.

From I, (5), and (6),

$$4(my^2 - lxy - mx^2 - p)^2 - 4k^4 - 4k^2(ly^2 + 4mxy - lx^2 - n) = 0$$

$$m^2y^4 - 2lmxy^3 + (l^2 - 2m^2)x^2y^2 - 2mpy^2 + 2lmx^3y + 2lpxy + m^2x^4 + 2mpx^2$$

$$+ p^2 - k^4 - k^2ly^2 - 4k^2mxy + k^2lx^2 + k^2n = 0,$$

which may be written

$$m^2x^4 + 2lmx^3y + (l^2 - 2m^2)x^2y^2 - 2lmxy^3 + m^2y^4 + (2mp + k^2l)x^2$$

$$- (4mk^2 - 2lp)xy - (2mp + k^2l)y^2 + p^2 - k^4 + k^2n = 0.$$

Therefore, by this transformation the straight lines parallel to the u-axis represented in Fig. (17.) by v_k, in the w-plane are transformed into the quartics, v'_k, in the z-plane, each of which always has two branches.

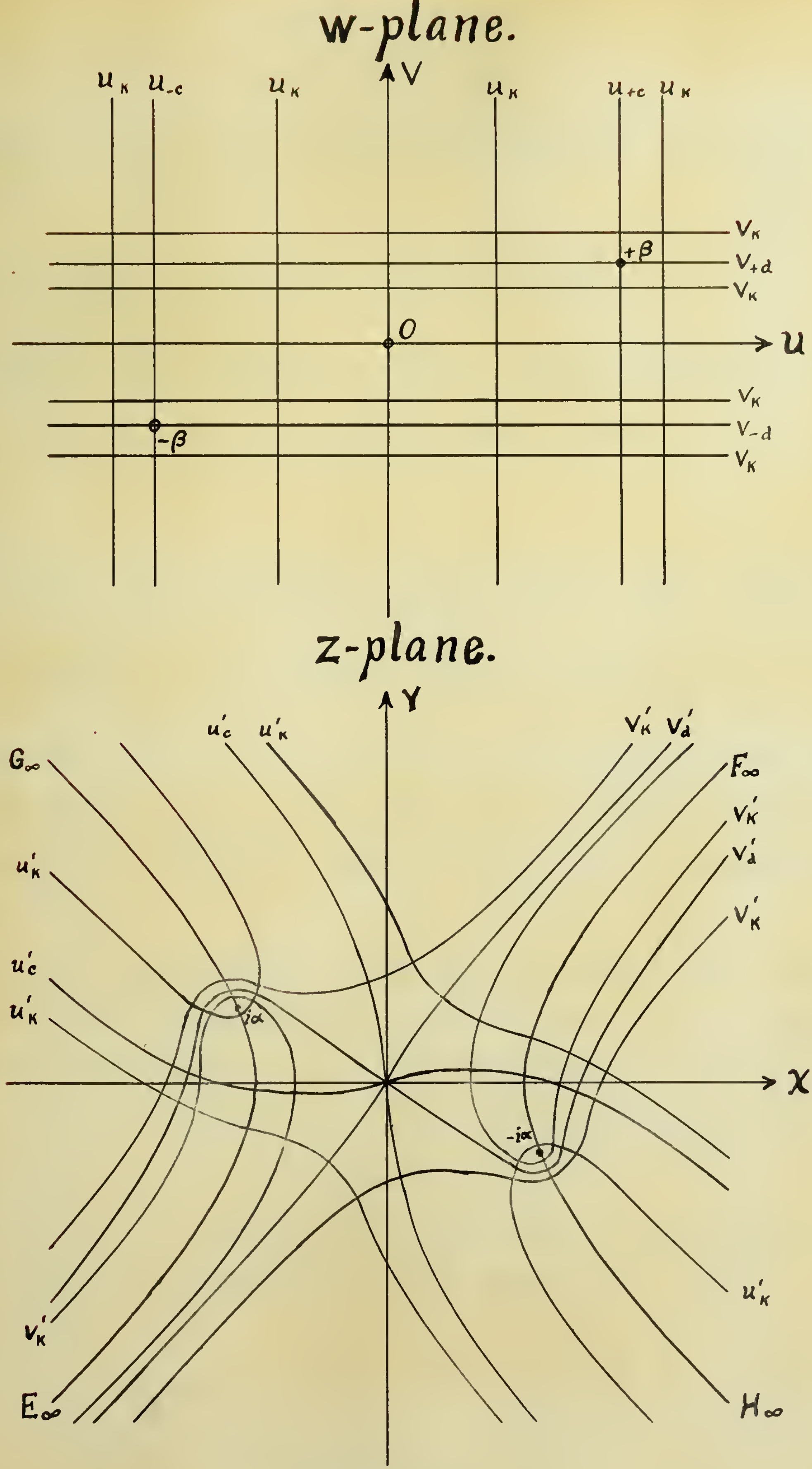


Fig. 17.

Second, we will let w describe the straight line $u = k$.

From II, (5), and (6),

$$4(my^2 - lxy - mx^2 - p)^2 - 4k^4 + 4k^2(ly^2 + 4mxy - lx^2 - n) = 0$$

$$m^2y^4 - 2lmxy^3 + (l^2 - 2m^2)x^2y^2 - 2mpy^2 + 2lmx^3y + 2lpxy + m^2x^4$$

$$+ 2mpx^2 + p^2 - k^4 + k^2ly^2 + 4mk^2xy - k^2lx^2 - k^2n = 0,$$

or

$$m^2x^4 + 2lmx^3y + (l^2 - 2m^2)x^2y^2 - 2lmxy^3 + m^2y^4 + (2mp - k^2l)x^2$$

$$+ (4mk^2 + 2lp)xy - (2mp - k^2l)y^2 + p^2 - k^4 - k^2n = 0.$$

Therefore by this transformation, the straight lines parallel to the v-axis, u_k, in the w-plane are transformed into the quartics u'_k, in the z-plane, each of which always has two branches, Fig. (17).

By this transformation, as in the preceding cases, in general, the curves u'_k are orthogonal to the curves v'_k, for the lines u_k are orthogonal to the lines v_k in the w-plane. However, in the case of the branch points we again have the isogonality breaking down.

To show this, consider the lines in the w-plane parallel to the axes which pass through the branch points $w = \pm \beta$.

We have

$$w = \pm \beta ,$$

$$u + iv = \pm (c + id).$$

Therefore $u = \pm c$,

and $v = \pm d$.

We will then let w describe either of the straight lines $v = \pm d$. From I, $B^2 - 4d^4 - 4d^2A = 0$.

From (5) and (6),

$$m^2x^4 + 2lmx^3y + (l^2 - 2m^2)x^2y^2 - 2lmxy^3 + m^2y^4 + (2mp + d^2l)x^2$$

$$- (4md^2 - 2lp)xy - (2mp + d^2l)y^2 + c^2d^2 - d^4 + d^4 - c^2d^2 = 0.$$

$$m^2x^4 + 2lmx^3y + (l^2 - 2m^2)x^2y^2 - 2lmxy^3 + m^2y^4 + (2mp + d^2l)x^2$$

$$- (4md^2 - 2lp)xy - (2mp + d^2l)y^2 = 0.$$

Therefore, by this transformation the straight lines v_{-d} and v_{+d}, passing through the branch points $-\beta$ and $+\beta$, respectively, are each transformed into the quartic, represented in Fig. (17.) by v_d^1, consisting of two branches which unite at the point $z = 0$.

Next, we will let w describe either of the straight lines $u = \pm c$.

From II, $B^2 - 4c^4 + 4c^2A = 0$.

From (5), and (6),

$$m^2x^4 + 2lmx^3y + (l^2 - 2m^2)x^2y^2 - 2lmxy^3 + m^2y^4 + (2mp - c^2l)x^2$$

$$+ (4mc^2 + 2lp)xy - (2mp - c^2l)y^2 + c^2d^2 - c^4 - c^2d^2 + c^4 = 0,$$

$$m^2x^4 + 2lmx^3y + (l^2 - 2m^2)x^2y^2 - 2lmxy^3 + m^2y^4 + (2mp - c^2l)x^2$$

$$+ (4mc^2 + 2lp)xy - (2mp - c^2l)y^2 = 0.$$

Therefore, by this transformation the straight lines u_{-c} and u_{+c}, passing through the branch points $-\beta$ and $+\beta$, respectively, are each transformed into the quartic represented in Fig. (17.) by u'_c, consisting of two branches which unite at the point $z = 0$.

In this function also, the two branches of v'_d are perpendicular to each other at $z = 0$, as also are the branches of u'_c.

Also the tangents to v'_d at $z = 0$ bisect the angles formed by the tangents to u'_c at $z = 0$ and conversely.

We may expect another irregularity in the conformal mapping of this function at the point $w = 0$. If we let w describe the straight line $v = 0$, we see from I and (6) that z describes the equilateral hyperbola,

$$my^2 - lxy - mx^2 - p = 0,$$

which passes through the points $+i\alpha$ and $-i\alpha$. Also if we let w describe the straight line $u = 0$, we see that z describes the same equilateral hyperbola,

$$my^2 - lxy - mx^2 - p = 0.$$

By an investigation similar to that used in the similar preceding cases, we see that the entire hyperbola is not described in each case, but rather that the line $v = 0$ maps into that part of the hyperbola from $+i\alpha$ to E_∞ and $-i\alpha$ to F_∞; also that the line $u = 0$ maps into that part from $+i\alpha$ to G_∞ and $-i\alpha$ to H_∞ (Fig.17).

Therefore, at the point $w = 0$, which maps into the two points $+i\alpha$ and $-i\alpha$ in the z-plane, we have the isogonality breaking down in the same way as in the similar preceding cases.

5. The Parabola.

Given the equation $w^2 = \alpha z$.

We have in this case the simple two valued function which is discussed in Harkness and Moreley's "Introduction to Analytic Functions", Example I, page 273, and also in "Functions of a Complex Variable" by E. J. Townsend.

From these discussions we see that the branch points in the z-plane are $z = 0$ and $z = \infty$.

For the discussion of this function we must consider a two-sheeted Riemann surface spread over the z-plane and one with but one sheet for the w-plane.

Since the branch points of this function are only in the z-plane we will discuss the mapping of the z-plane upon the w-plane by means of the given function, $z = \frac{w^2}{\alpha}$.

We then have

$$x + iy = \frac{u^2 + 2iuv - v^2}{a + ib}.$$

$$x + iy = \frac{(u^2 + 2iuv - v^2)(a - ib)}{a^2 + b^2},$$

$$= \frac{au^2 + 2iauv - av^2 - ibu^2 + 2buv + ibv^2}{a^2 + b^2}.$$

Equating the real parts,

$$x = \frac{au^2 - av^2 + 2buv}{a^2 + b^2};$$

and the imaginary parts,

$$y = \frac{2auv - bu^2 + bv^2}{a^2 + b^2}.$$

First, we will let z describe the straight line $x = +k$, in which k is a positive number.

Then
$$\frac{au^2 - av^2 + 2buv}{a^2 + b^2} = k$$

$$au^2 - av^2 + 2buv = k(a^2 + b^2)$$

Therefore, by this transformation the lines, x_{+k}, parallel to the y-axis and having positive values of x, are transformed into the equilateral hyperbolas, x'_{+k}, whose transverse axis has the slope $\dfrac{\sqrt{a^2 + b^2} - a}{b}$, (Fig. 18).

Second, let z describe the straight line $x = -k$.

Then
$$au^2 - av^2 + 2buv = -k(a^2 + b^2),$$

or
$$av^2 - au^2 - 2buv = k(a^2 + b^2).$$

Therefore, by this transformation the lines, x_{-k}, parallel to the y-axis and having negative values of x, are transformed into the equilateral hyperbolas, x'_{-k}, in the w-plane, whose transverse axis has the slope $\dfrac{-\sqrt{a^2 + b^2} - a}{b}$ (Fig. 18).

Third, let z describe the straight line $y = +k$.

Then
$$\frac{2auv - bu^2 + bv^2}{a^2 + b^2} = k,$$
$$bv^2 - bu^2 + 2auv = k(a^2 + b^2)$$

Therefore, by this transformation, the lines, y_{+k}, parallel to the x-axis and having positive values of y, are transformed into the equilateral hyperbolas, y'_{+k}, in the w-plane, whose transverse

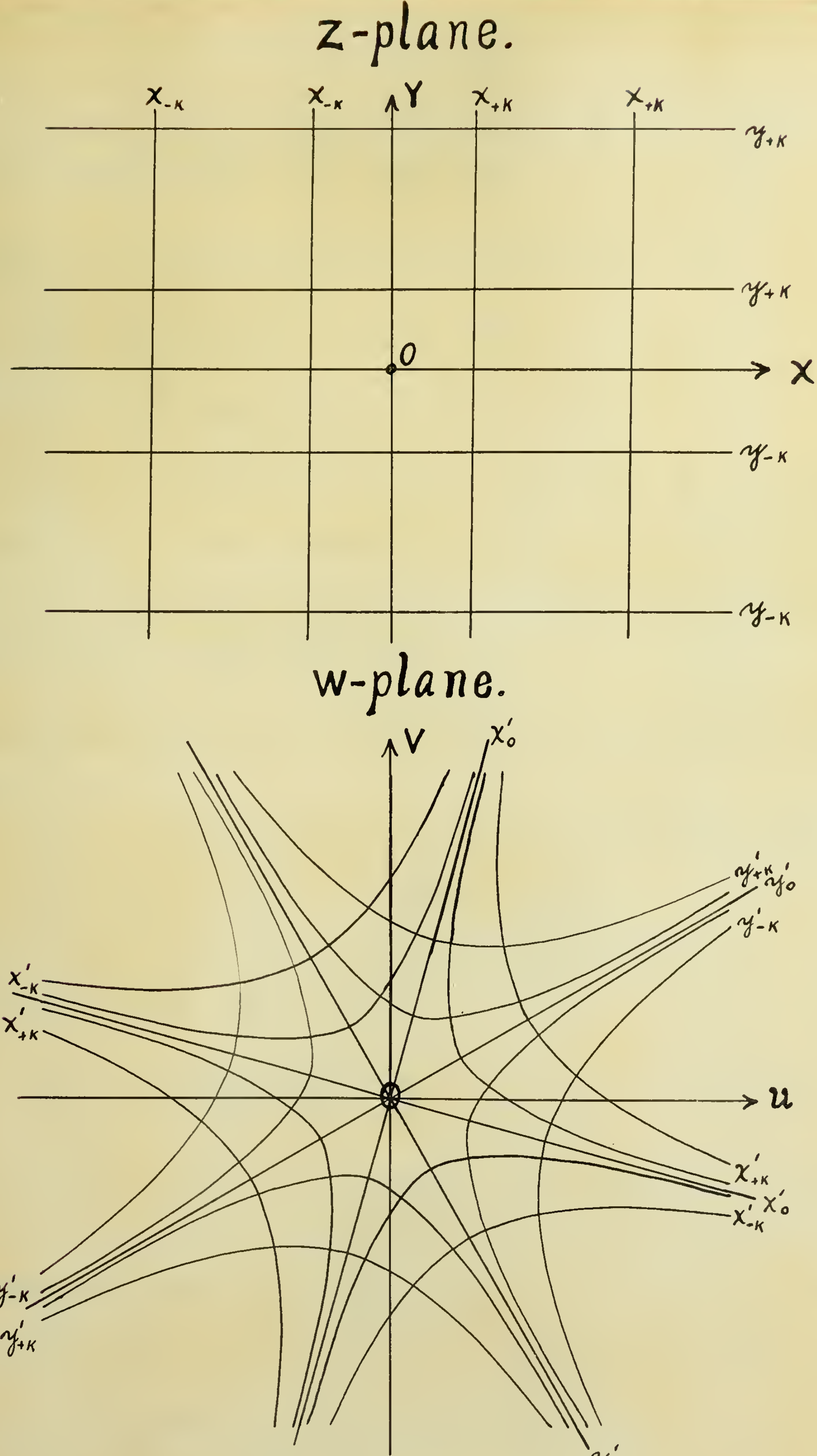

z-plane.
X_{-K}
X_{-K}
Y
X_{+K}
X_{+K}
y_{+K}
y_{+K}
O
X
y_{-K}
y_{-K}
w-plane.
V
x'_0
y'_{+K}
y'_0
y'_{-K}
x'_{-K}
x'_{+K}
u
x'_{+K}
x'_0
x'_{-K}
y'_{-K}
y'_{+K}
y'_0

Fig. 18.

axis has the slope $\frac{b - \sqrt{a^2 + b^2}}{a}$ (Fig. 18).

Fourth, let z describe the straight line $y = -k$,

Then
$$bv^2 - bu^2 + 2auv = -k(a^2 + b^2),$$

or
$$bu^2 - bv^2 - 2auv = k(a^2 + b^2).$$

Therefore, by this transformation, the lines, y_{-k}, parallel to the x-axis and having negative values of y are transformed into the equilateral hyperbolas, y'_{-k}, in the w-plane, whose transverse axis has the slope $\frac{b + \sqrt{a^2 + b^2}}{a}$ (Fig. 18).

We know that one of the branch points of this function in the z-plane is $z = 0$.

Therefore, we will discuss the mapping of the two lines $x = 0$ and $y = 0$, which pass through this branch point.

We will then let z describe the straight line $x = 0$.

Then $au^2 + 2buv - av^2 = 0$,

or
$$v = \left(\frac{b \pm \sqrt{a^2 + b^2}}{a} \right) u.$$

Therefore, by this transformation, the y-axis, x_0, is transformed into the degenerate equilateral hyperbola, which consists of two orthogonal lines, x'_0 intersecting at $w = 0$ and having the slopes $\frac{b + \sqrt{a^2 + b^2}}{a}$ and $\frac{b - \sqrt{a^2 + b^2}}{a}$, respectively (Fig. 18).

Then, let z describe the straight line $y = 0$.

Then
$$bv^2 + 2auv - bu^2 = 0,$$

or
$$v = \left(\frac{-a \pm \sqrt{a^2 + b^2}}{b}\right) u.$$

Therefore, by this transformation, the x-axis, y_o, is transformed into the degenerate equilateral hyperbola, which consists of two orthogonal lines, y'_o, having the slopes $\frac{-a+\sqrt{a^2+b^2}}{b}$ and $\frac{-a-\sqrt{a^2+b^2}}{b}$, respectively and intersecting at $w = 0$ (Fig. 18).

We also know that the other branch point of this function is $z = \infty$, which is transformed into the point $w = \infty$. Therefore, to discuss the function at these points, we must make the transformation $z' = \frac{1}{z}$ and $w' = \frac{1}{w}$ and make the discussion at the points $z' = 0$ and therefore $w' = 0$.

Given $w^2 = \alpha z$.

Then $$\frac{1}{w'^2} = \frac{\alpha}{z'},$$

and $$z' = \alpha w'^2.$$

$$x' + iy' = (a + ib)(u'^2 + 2iu'v' - v'^2),$$

$$x' + iy' = au'^2 + 2iau'v' - av'^2 + ibu'^2 - 2bu'v' - ibv'^2.$$

Equating the real parts,

$$x' = au'^2 - av'^2 - 2bu'v';$$

and the imaginary parts,

$$y' = 2au'v' + bu'^2 - bv'^2.$$

Now we will let z' describe the straight line $x' = 0$.

Then $au'^2 - av'^2 - 2bu'v' = 0$, $av'^2 - au'^2 + 2bu'v' = 0$,

and $$v' = \left(\frac{-b \pm \sqrt{a^2+b^2}}{a}\right) u'.$$

Therefore the line $x' = 0$ is transformed into the two orthogonal lines, passing through $w' = 0$ and having the slopes $\frac{-b+\sqrt{a^2+b^2}}{a}$ and $\frac{-b-\sqrt{a^2+b^2}}{a}$, respectively.

Also let z' describe the straight line $y' = 0$.

Then

$$bu'^2 - bv'^2 + 2au'v' = 0,$$

$$v' = \left(\frac{a \pm \sqrt{a^2+b^2}}{b}\right) u'.$$

Therefore we see that the line $y' = 0$ is transformed into the two orthogonal lines passing through $w' = 0$ and having the slopes $\frac{a+\sqrt{a^2+b^2}}{b}$ and $\frac{a-\sqrt{a^2+b^2}}{b}$, respectively.

From the discussion made concerning the branch points, we see that the general laws are true in regard to this case. The line $x = 0$, passing through the branch point $z = 0$, is transformed by this function into two orthogonal branches in the w-plane, intersecting at $w = 0$. Also the line $y = 0$, passing through $z = 0$ is transformed into two orthogonal branches which also intersect at $w = 0$ and furthermore bisect the angles made by the x_0' lines. We may also notice from the discussion of the branch point $z = \infty$ that the same laws are true in regard to this point.

6. Investigation and Representation of the Imaginary Domains of Conics in the Complex Plane.

An important special case of the preceding discussion is obtained by assuming for the domain of one of the variables, for instance the variable Z, the domain of all real numbers. This leads to the ordinary theory of conics with the addition of imaginary domains. In this section I shall take up only that part of the theory connected with the imaginary domains.

<u>The Circle.</u>

Given the equation of the system of circles,

$$x^2 + y^2 = a^2$$

When $x^2 < a^2$ and $y^2 < a^2$, this equation represents a system of circles with both variables in real domains. But if $x^2 > a^2$ or $y^2 > a^2$, we get imaginary values for y or x respectively.

First, let $x^2 > a^2$.

Then $y = \pm i\sqrt{x^2 - a^2}$.

Transforming this equation into the complex plane, by means of the equations $x = u$ and $y = i\,v$,

we get $v = \pm\sqrt{u^2 - a^2}$, which may be written in the form

$$u^2 - v^2 - a^2 = 0$$

or $$\frac{u^2}{a^2} - \frac{v^2}{a^2} = 1.$$

This is the equation of a system of equilateral hyperbolas, whose axis coincides with the u - axis (Fig. 19.).

Second, let $y^2 > a^2$

Then $x = \pm i\sqrt{y^2 - a^2}$.

Transforming this equation into the complex plane by means of the equations $x = i\,u$ and $y = v$,

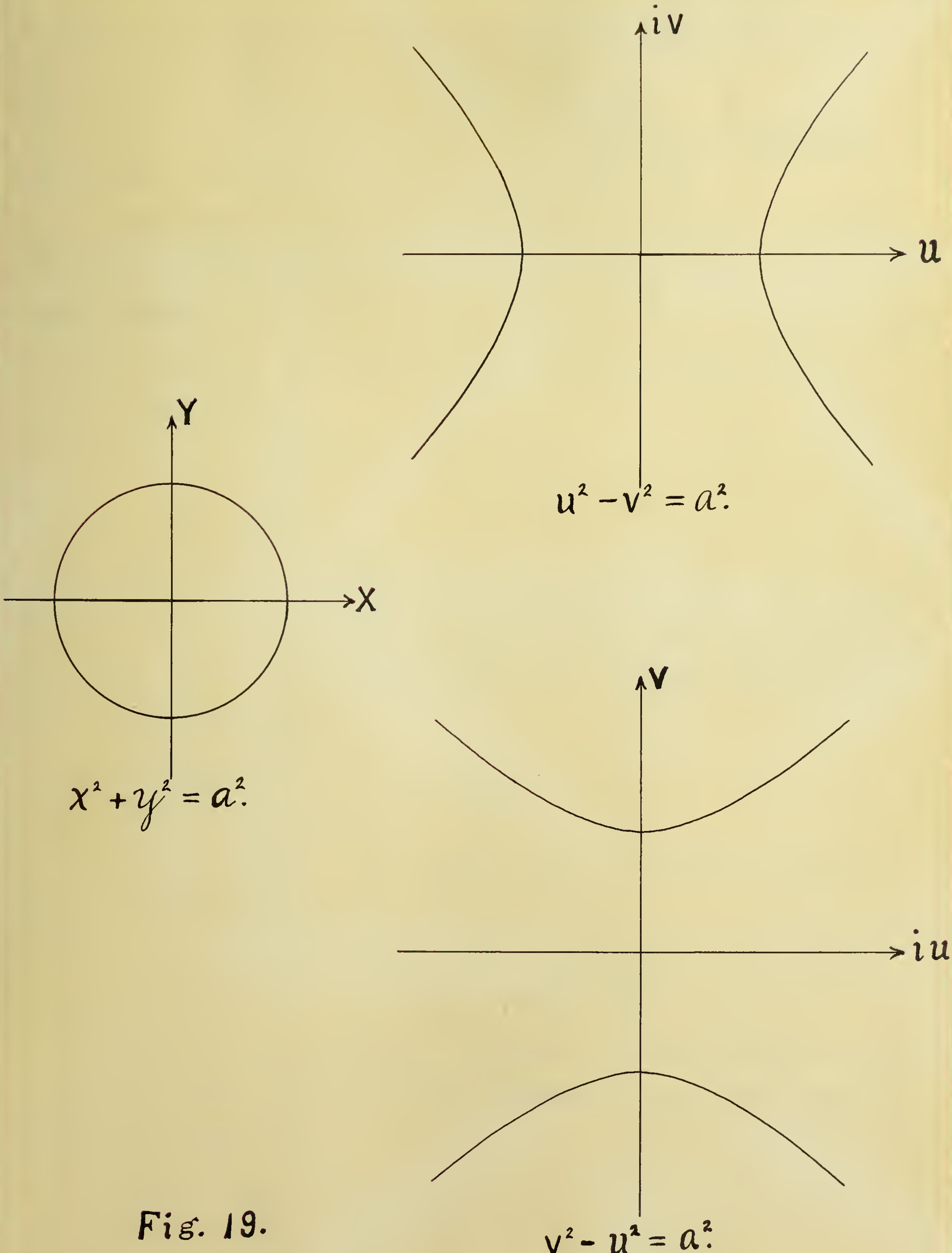


Fig. 19.

we get $u = \pm\sqrt{v^2 - a^2}$, which may be written in the form

$$v^2 - u^2 - a^2 = 0 \quad \text{or} \quad \frac{v^2}{a^2} - \frac{u^2}{a^2} = 1.$$

This is the equation of a system of equilateral hyperbolas, whose axis coincides with the v - axis (Fig. 19).

Therefore, we see that the imaginary domains of a circle in the real plane, map into the two equilateral hyperbolas in the complex plane, one on each of the coordinate axes. Also, the axes of the hyperbolas are equal to the diameter of the circle.

<u>The Ellipse.</u>

Given the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

When $x^2 < a^2$ and $y^2 < b^2$, this is the equation of a system of real ellipses, with their centers at the origin and foci on the x - axis. However, if $x^2 > a^2$ or $y^2 > b^2$, we get imaginary values for y or x, respectively.

First, let $x^2 > a^2$.

Then $$y = \pm \frac{b}{a} i \sqrt{x^2 - a^2}.$$

By transforming this equation into the complex plane, by means of the equations $x = u$ and $y = i\,v$,

we get $v = \pm \frac{b}{a}\sqrt{u^2 - a^2}$, which may be written in the form

$$a^2v^2 - b^2u^2 + a^2b^2 = 0 \quad \text{or} \quad \frac{u^2}{a^2} - \frac{v^2}{b^2} = 1.$$

This is the equation of a system of hyperbolas with its foci on the u-axis, where 2a is the transverse axis and 2b the conjugate axis (Fig. 20).

Second, let $y^2 > b^2$.

Then $x = \pm \frac{a}{b}\, i \sqrt{y^2 - b^2}$.

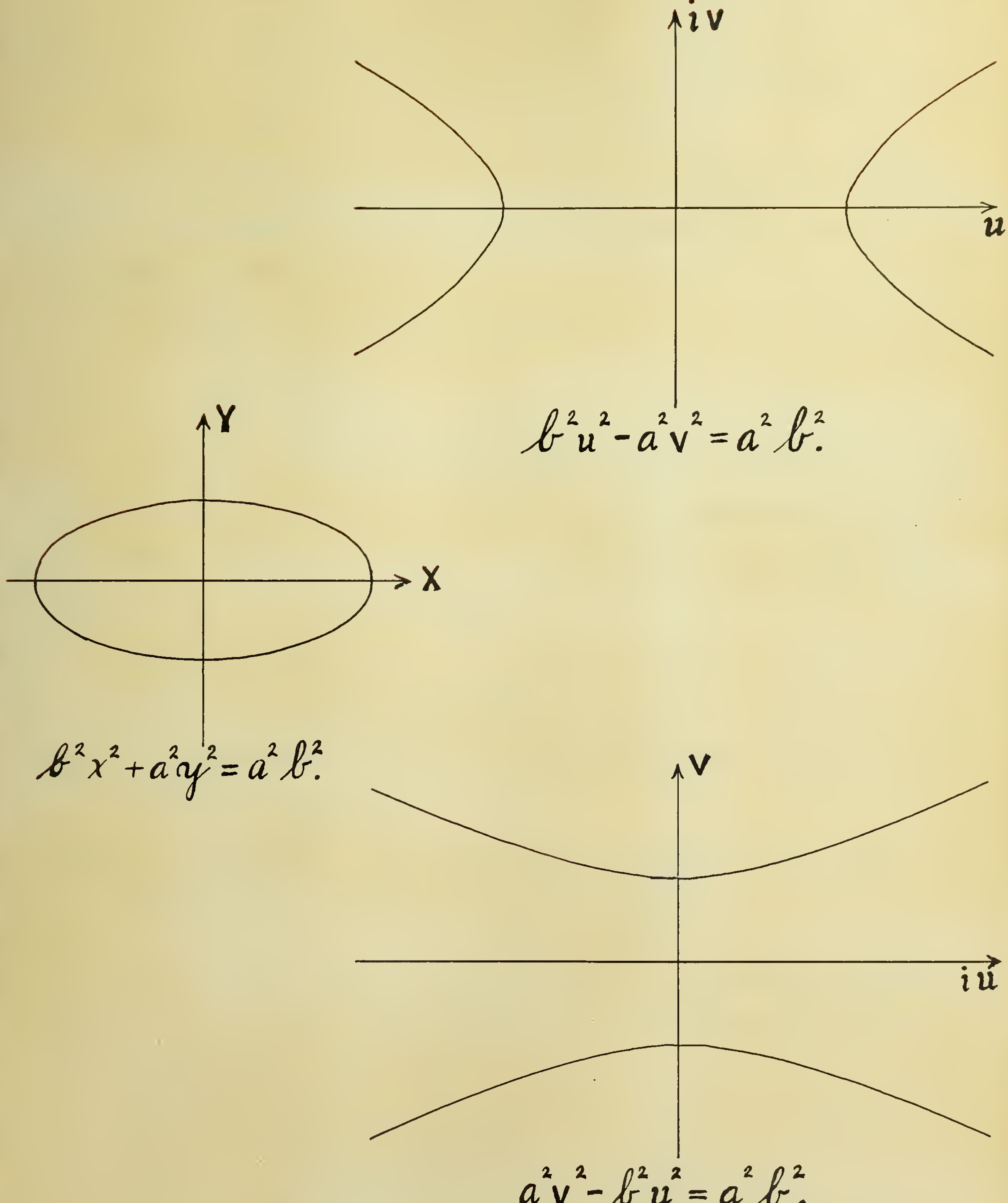


Fig. 20.

By transforming this equation into the complex plane, by means of the equations $x = i\,u$ and $y = v$,

we get $u = \pm \frac{a}{b}\sqrt{v^2 - b^2}$, which may be written in the form

$$a^2v^2 - b^2u^2 - a^2b^2 = 0, \text{ or } \frac{v^2}{b^2} - \frac{u^2}{a^2} = 1.$$

This is the equation of a system of hyperbolas with its foci on the v-axis, where 2 b is the transverse axis and 2 a the conjugate axis (Fig. 20).

We will then take the equation

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

When $x^2 < b^2$ and $y^2 < a^2$, this equation represents a system of ellipses, with their centers at the origin and foci on the y-axis. However, if $x^2 > b^2$ or $y^2 > a^2$, we get imaginary values for y or x, respectively.

First, let $x^2 > b^2$.

Then $y = \pm \frac{a}{b}\, i \sqrt{x^2 - b^2}$

By transforming this equation by means of the equations

$x = u$ and $y = i\,v$,

we get $v = \pm \frac{a}{b}\sqrt{u^2 - b^2}$, which may be written in the form

$$a^2u^2 - b^2v^2 - a^2b^2 = 0 \text{ or } \frac{u^2}{b^2} - \frac{v^2}{a^2} = 1.$$

This is the equation of a system of hyperbolas with its foci on the u - axis, where 2 b is the transverse axis and 2 a the conjugate axis. (Fig. 21).

Second, let $y^2 > a^2$.

Then $x = \pm \frac{b}{a}\, i \sqrt{y^2 - a^2}$

By transforming this equation by means of the equations

$x = i\,u$ and $y = v$,

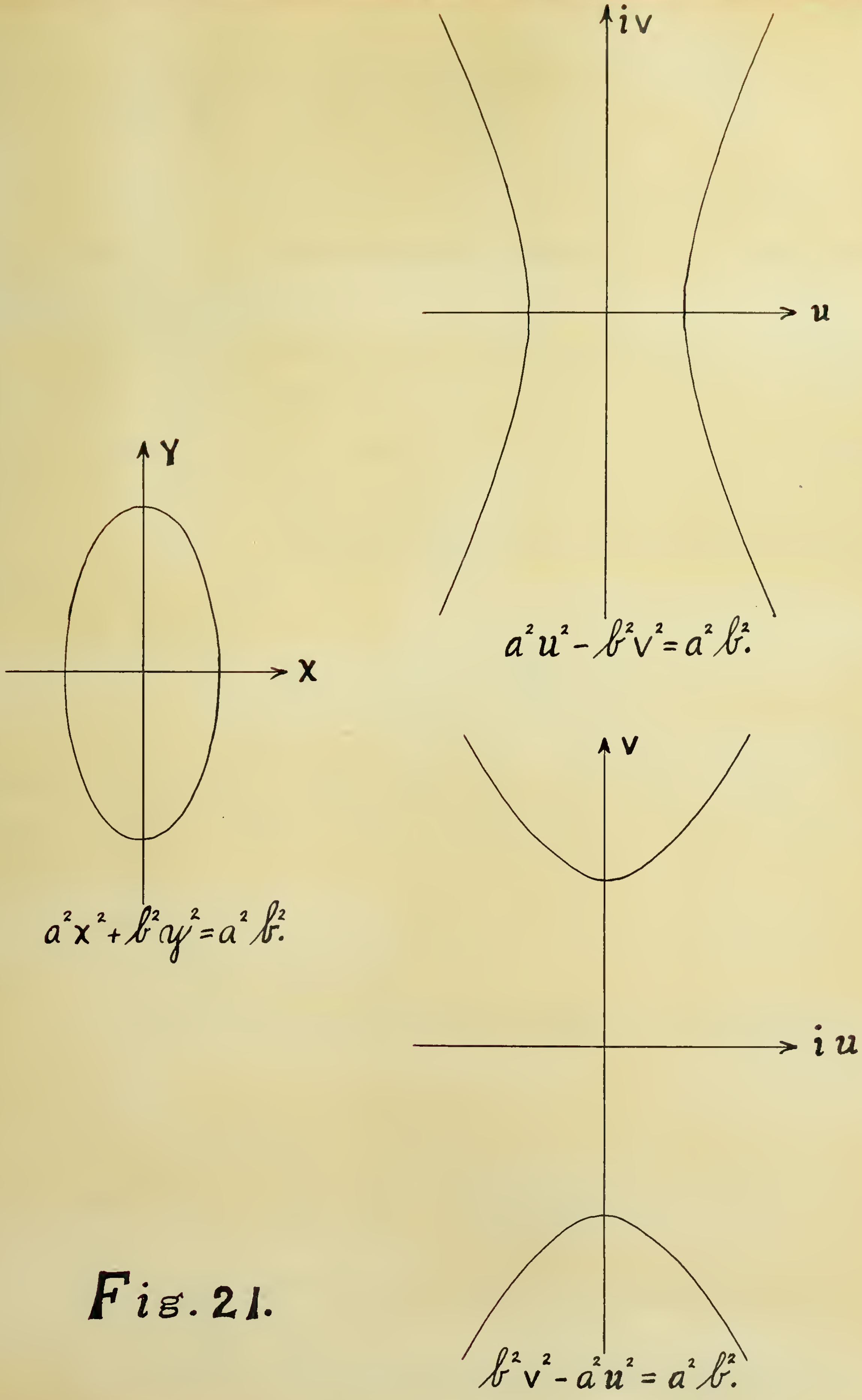


Fig. 21.

we get $u = \pm \frac{b}{a}\sqrt{v^2 - a^2}$, which may be written in the form

$$b^2v^2 - a^2u^2 - a^2b^2 = 0\,, \quad \text{or } \frac{v^2}{a^2} - \frac{u^2}{b^2} = 1.$$

This is the equation of a system of hyperbolas with its foci on the v - axis, where 2 a is the transverse axis and 2 b the conjugate axis. (Fig. 21).

Therefore, we see that the imaginary domains of an ellipse in the real plane map into two hyperbolas in the complex plane, one of which is on the u - axis and has its transverse axis equal to the axis of the ellipse on the x - axis. The other is on the v - axis and has its transverse axis equal to the axis of the ellipse on the y - axis.

The Hyperbola.

We will first take the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

When $x^2 > a^2$ and y varies from $-\infty$ to $+\infty$, this equation represents a system of real hyperbolas, with their centers at the origin and foci on the x - axis. But if $x^2 < a^2$, we get imaginary values for y.

Let $x^2 < a^2$.

Then $\qquad y = \pm \frac{b}{a}\, i \sqrt{a^2 - x^2}.$

By transforming this equation by means of the equations

$$x = u$$

and $\qquad y = i\,v,$

we get $v = \pm \frac{b}{a}\sqrt{a^2 - u^2}$, which may be written in the form

$$a^2v^2 + b^2u^2 - a^2b^2 = 0, \quad \text{or} \quad \frac{u^2}{a^2} + \frac{v^2}{b^2} = 1.$$

This is the equation of a system of ellipses with its foci on the u - axis (Fig. 22).

In a similar manner, by taking the equation $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$,

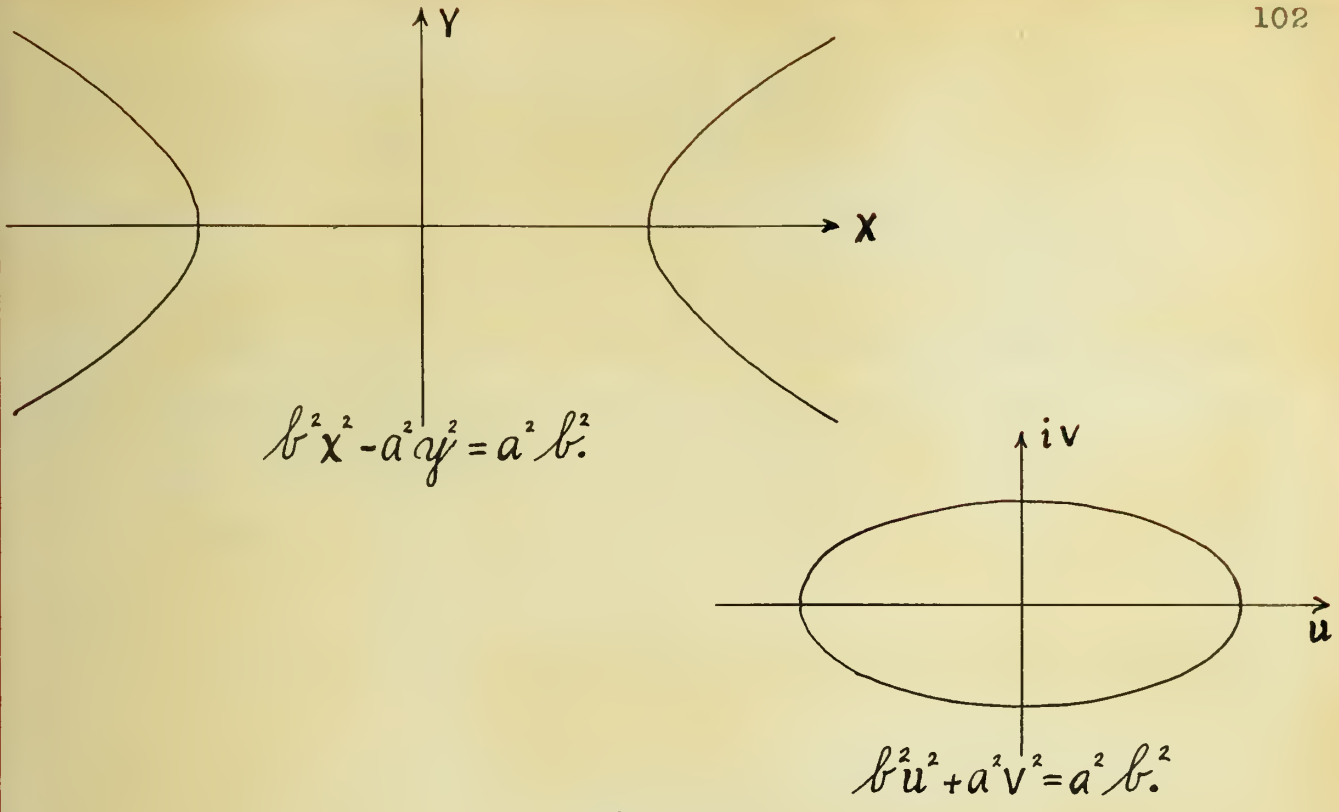

Y
X
$b^2x^2 - a^2y^2 = a^2b^2.$
iv
u
$b^2u^2 + a^2v^2 = a^2b^2.$

Fig. 22.

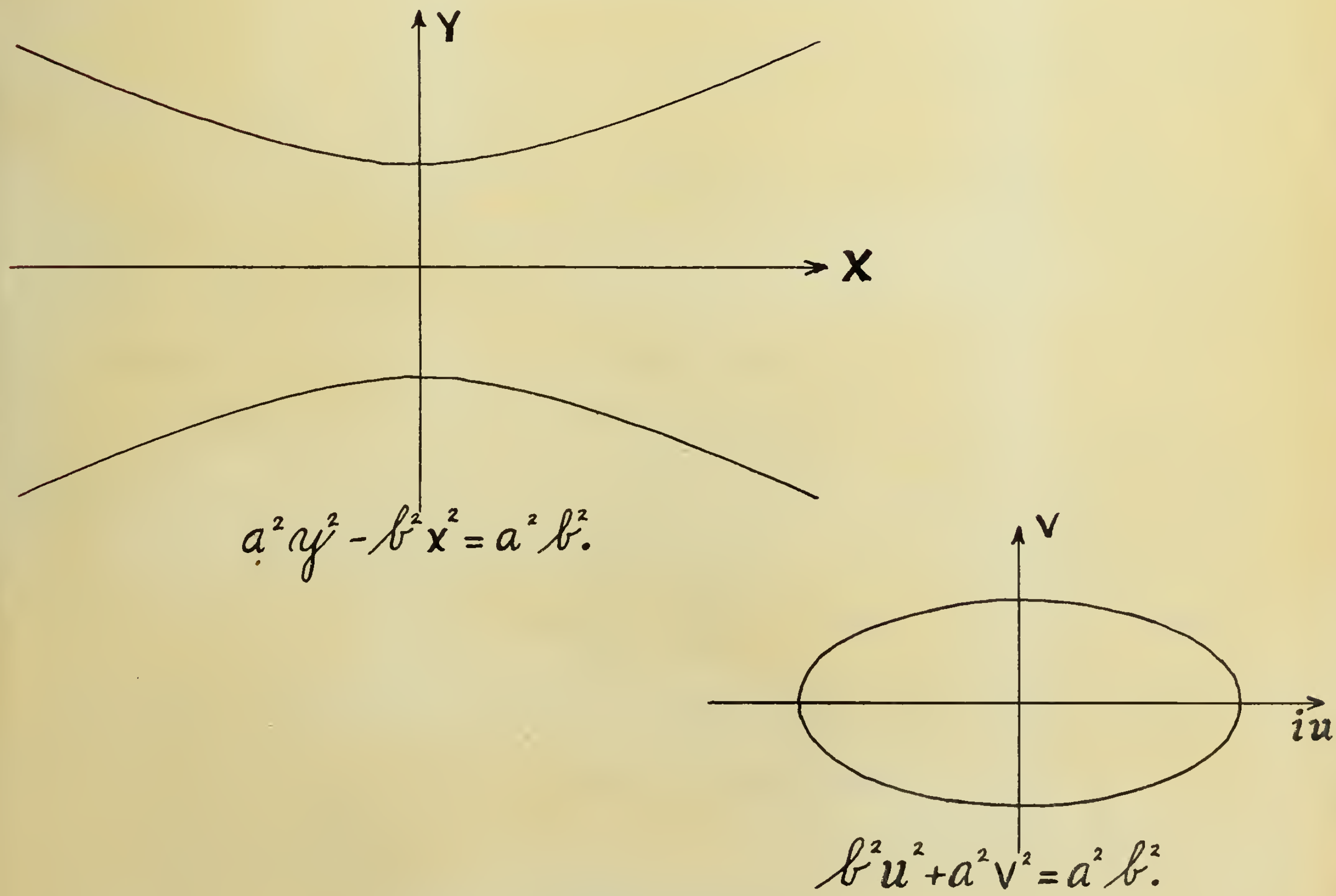

Y
X
$a^2y^2 - b^2x^2 = a^2b^2.$
v
iu
$b^2u^2 + a^2v^2 = a^2b^2.$

Fig. 23.

letting $x^2 < b^2$, and transforming by the same equations, we get

$a^2u^2 + b^2v^2 - a^2b^2 = 0$, which is the equation of a system of ellipses with its foci on the v - axis (Fig. 24).

Then take the equation $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

When $y^2 > b^2$, this equation represents a system of real hyperbolas with the origin as the common center and foci on the y - axis. But if $y^2 < b^2$, we get imaginary values for x.

Let $y^2 < b^2$.

Then $$x = \pm \frac{a}{b} i \sqrt{b^2 - y^2}.$$

By transforming this equation by means of the equations

$x = i u$, and $y = v$,

we get $u = \pm \frac{a}{b} \sqrt{b^2 - v^2}$, which may be written in the form

$$b^2u^2 + a^2v^2 - a^2b^2 = 0, \quad \text{or} \quad \frac{u^2}{a^2} + \frac{v^2}{b^2} = 1.$$

This is the equation of a system of ellipses with its foci on the u - axis (Fig. 23).

In a similar manner by taking the equation $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, letting $y^2 < a^2$, and transforming by means of the same equations, we get $a^2u^2 + b^2v^2 - a^2b^2 = 0$, which is the equation of a system of ellipses with its foci on the v-axis (Fig. 25).

Therefore, we see that the imaginary domains of the two hyperbolas in the real plane, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, map into like ellipses in the complex plane. Also the imaginary domains of the two hyperbolas $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ map into like ellipses. However, in both cases the like ellipses have the imaginary and real axes interchanged.

<u>The Parabola.</u>

First, take the equation $y^2 = 4 p x$.

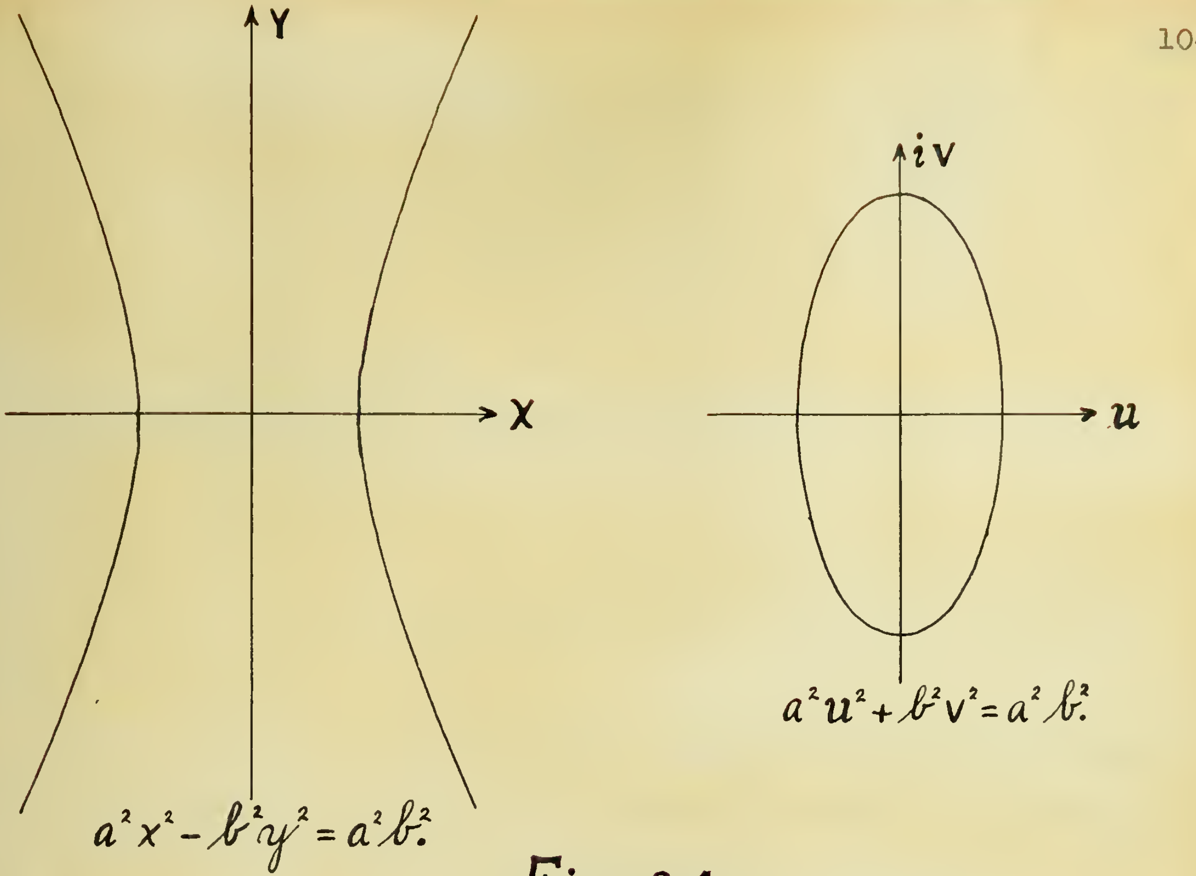

Y
X
$a^2x^2 - b^2y^2 = a^2b^2.$
iv
u
$a^2u^2 + b^2v^2 = a^2b^2.$

Fig. 24.

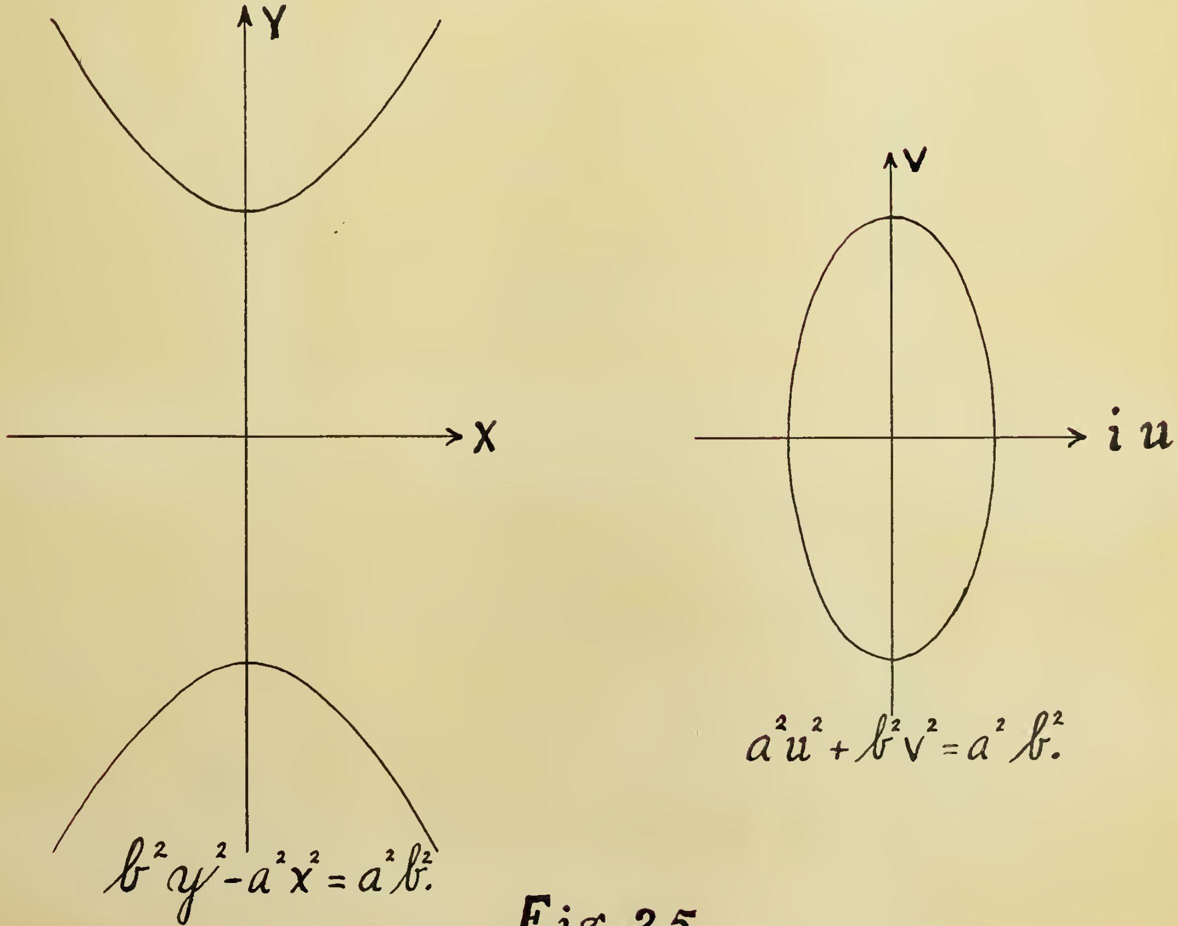

Y
X
$b^2y^2 - a^2x^2 = a^2b^2.$
v
iu
$a^2u^2 + b^2v^2 = a^2b^2.$

Fig. 25.

When $p > o$ and $x > o$, this is the equation of a system of parabolas, with the vertex at the origin and foci on the positive x - axis. But if $x < 0$, we get imaginary values for y.

So, let $x < 0$.

Then $$y = \pm 2 i \sqrt{p x}.$$

By transforming this equation by means of the equations

$x = u$ and $y = i v$,

we get $v = \pm 2\sqrt{p u}$ or $v^2 = 4 p u$, which is also a system of parabolas with the foci on the u - axis, for positive values of u (Fig. 26).

When $p < 0$ and $x < 0$, the given equation represents a system of parabolas with the vertex at the origin and foci on the negative x - axis. But if $x > 0$, we get imaginary values for y.

Therefore, let $x > 0$.

Then $$y = \pm 2 i \sqrt{p x}.$$

Transforming this equation by means of the equations

$x = u$ and $y = i v$, we get $v = \pm 2\sqrt{p u}$, or $v^2 = 4 p u$, which is a system of parabolas with the foci on the u - axis, for negative values of u, for p is negative (Fig. 26).

Second, take the equation

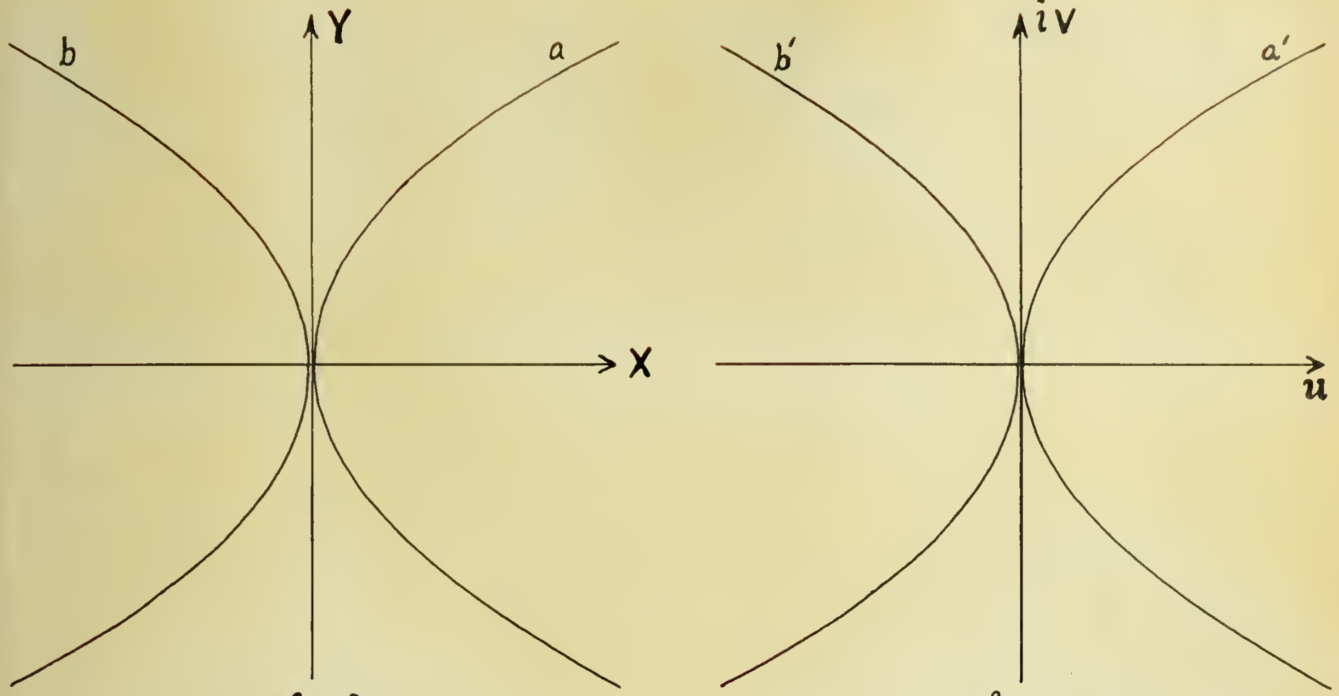
$x^2 = 4 p y$.

When $p > 0$ and $y > 0$, this equation represents a system of parabolas, with the vertex at the origin and the foci on the positive y - axis. But if $y < 0$, we get imaginary values for x.

Then, let $y < 0$.

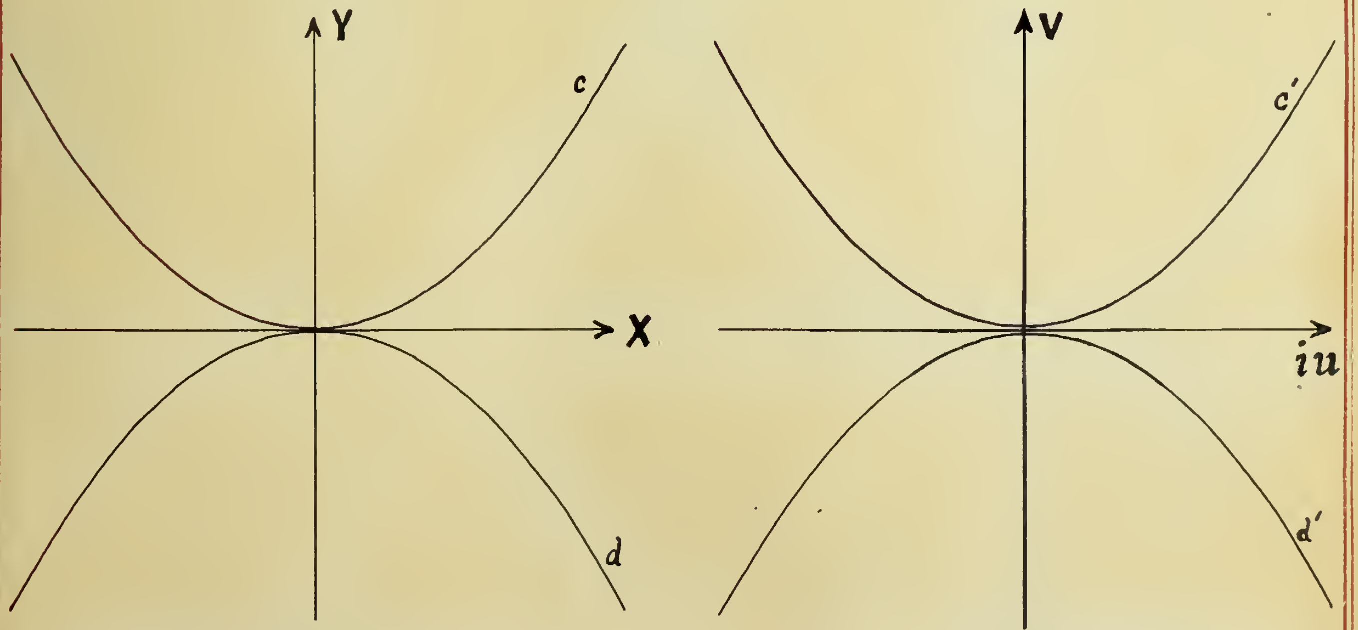
Then $$x = \pm 2 i \sqrt{p y}.$$

Transforming this equation by means of the equations



$y^2 = 4px.$ $v^2 = 4pu$

Fig. 26.



$x^2 = 4py.$ $u^2 = 4pv.$

Fig. 27.

$x = i u$ and $y = v$, we get $u = \pm 2\sqrt{p v}$ or $u^2 = 4 p v$, which is the equation of a system of parabolas with the foci on the v - axis for positive values of v. (Fig. 27).

When $p < 0$ and $y < 0$, the given equation represents a system of parabolas with the vertex at the origin and the foci on the negative y - axis. But if $y > 0$, we get imaginary values for x.

So, let $y > 0$.

Then $$x = \pm 2 i \sqrt{p y}.$$

Transforming this equation by means of the equations

$x = i u$ and $y = v$, we get $u = \pm 2\sqrt{p v}$ or $u^2 = 4 p v$, which is the equation of a system of parabolas with the foci on the v - axis for negative values of v, for in this case p is negative (Fig. 27).

Therefore we see in each case of the parabola, that the imaginary domains of the real plane are transformed into the complex plane in such a way that the figures in both planes are exactly alike.

CHAPTER IV.

A SHORT HISTORICAL ACCOUNT OF THE DEVELOPMENT MADE WITH IMAGINARY GEOMETRICAL ELEMENTS.

Imaginary numbers were discovered as roots of equations many centuries ago, but were thought to be useless until in recent years they were given a geometrical representation.

As early as 1545, Cardan in his celebrated memoir, "Ars Magna", discussed the imaginary roots of equations and proved that they always occur in pairs, but declined to commit himself to any explanation as to their meaning.[*]

By 1572, Raphael Bombelli[*] had advanced the discussion somewhat, but it was not until almost two centuries later, in 1750-51, that Küln of Danzig proposed a geometric interpretation of the imaginary number $a\sqrt{-1}$.[1]

In 1797, Caspar Wessel[2] published a memoir "On the Analytic Representation of Direction", in which he showed that he was familiar with the three present methods of representing the complex number, $a + b\sqrt{-1}$, $r(\cos\phi + \sqrt{-1}\sin\phi)$ and $re^{\phi\sqrt{-1}}$.

The common method used to-day for representing the complex number $a + b\sqrt{-1}$ in the complex plane was fully presented in 1806 by Jean-Robert Argand[3] in his famous memoir, "Essai sur une maniére de représenter les quantités imaginaires dans les constructions géométriques."

In 1799 Gauss[2] indicated that he was in possession of a me-

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[*] A Short History of Mathematics; W. W. R. Ball, p. 199
A History of Mathematics; F. Cajori, p 146.

[1&3] "A History of Mathematics"; F. Cajori, p 317.

[2] "A Chapter on the History of Mathematics"; W. W. Beman. Proceedings of the American Association for the Advancement of Science. Vol 46. p. 33.

[3] Chasles, Rapport sur les Progrés de la Geometrie, p. 60.

thod of dealing with quantities of the form $a + b\sqrt{-1}$, but its fuller exposition was deferred until 1831.

However, pure imaginary points and straight lines could not be represented by this method.

In 1822, Poncelet[1] discovered the imaginary circular points of the plane in the following way. He found that the system of hyperbolas, whose asymptotes are two systems of parallel lines, cuts the line at infinity in two real points. Likewise, the system of ellipses whose axes are proportional and therefore whose asymptotes are two systems of imaginary parallel lines, cuts the line at infinity in two imaginary points. Since all circles constitute a special case of such a system of ellipses, the two imaginary points of their intersections with the line at infinity are defined by Poncelet as the Circular Points of the plane.

Following this, we have the development of the theory for representing imaginary points and straight lines by Cauchy[2] in 1847.

In 1856, Von Staudt[3] developed a method of representing imaginary elements geometrically by means of the theory of involution.

The method of Cauchy was further developed by M. E. Laguerre[4] in 1870 in the following way:-

He first developed the idea of the imaginary lines, which he

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[1] Traité des Propriétés Projectives des Figures. Second Edition, Vol. I, p. 47 - 48.
[2] "Sur les quantites geometriques". Oeuvres de A. Cauchy. Series, I, Vol. XI p. 152.
[3] "Beitrage zur Geometrie de Lage", Von Standt, p. 76.
[4] Oeuvres de Laguerre, Paris, 1905. Vol. II. Ps. 88 - 108.

called the Isotropic lines of the plane. Starting with the equation of the circle $(x - a)^2 + (y - b)^2 = r^2$, if $r = 0$, he got

$$(x - a)^2 + (y - b)^2 = 0,$$

which he factored, getting the two equations

$$(x - a) + i(y - b) = 0$$

and $$(x - a) - i(y - b) = 0,$$

which are the equations of two straight lines passing through the point (a, b) and having the slopes $+i$ and $-i$, respectively. As (a, b) may take any position in the plane, these equations represent the two systems of parallel imaginary lines, called by Laguerre, Isotropic Lines. The system with the slope $+i$, he called the first system, and that with slope $-i$, the second system. The first system of lines is concurrent with the line at infinity at the imaginary point I_1 and the second system is concurrent at infinity at I_2. Laguerre defined these points I_1 and I_2 as the Ombilics or the Circular points discovered by Poncelet.

Laguerre proved that each imaginary straight line contains one and only one real point, which offers him the following unique method of representing an imaginary point:

Draw an isotropic line of each system through the given imaginary pointA. Call the real point on the line of the first system a and that on the line of the second system a', then the line aa' which is real, is defined as the Representative Segment of the imaginary point A. This is a unique representation for there is a one to one correspondence between each representative segment and each imaginary point of the plane and also any transformation of axes

leaves the point and its representative segment unchanged. The points a and a' are defined as the origin and extremity of the representative segment, respectively. If the point A is real, these two points and the entire segment coincide with A.

Laguerre also developed a system of Isotropic Coordinates as follows: Take the line $O\omega$ as the axis; draw through the point A, the two isotropic lines and call the intersection of the one of the first system with $O\omega$, α and that of the one of the second system with $O\omega$, β. Then the isotropic coordinates of A are $O\alpha = u$ and $O\beta = w$.

If we now take a system of rectangular coordinates such that $O\omega$ is the x - axis and a perpendicular through O is the y - axis and call the rectangular coordinates of A, x and y, we get

$u = x + iy$ and $w = x - iy$.

Laguerre defined two points as conjugate imaginary when their coordinates are respectively conjugate imaginary numbers. Also two straight lines are conjugate imaginary when one can be changed to the other by substituting +i for -i in its equation and conversely.

Therefore if we have given the imaginary point A and its representative segment a a', the conjugate imaginary of A a is the isotropic of the second system through a and that of A a' is the isotropic of the first system throuth a'. Also the point which is conjugate imaginary to A is A', the intersection of the two conjugate imaginary lines of A a and Aa' and is represented by the segment a'a.

Laguerre also proved that AA' is real and is the perpendicular bisector of the segment a a'.

The square of the distance between two imaginary points

A = (a a') and B = (b b') was found by Laguerre to be an imaginary number, the modulus of which is the product of the distances ab and a'b' and the amplitude of which is the positive angle which ab makes with a'b'. That is $\overline{AB}^2 = ab.a'b'.e^{\mu i}$.

Laguerre obtained the following results, either directly concerning imaginaries or by their application.

Considering the imaginary points lying on a curve represented in isotropic coordinates by f(u,v) = 0, for each value of w, which determines a certain origin of a representative segment, there will be in general as many different values of w as the degree of the equation and therefore as many different representative segments having different extremities but all having the same origin.

Also if the curve is real, for every point which it contains, it also contains its conjugate imaginary.

Considering any number of imaginary points lying on a straight line, their representative segments are such that the origins form a polygon which is similar to that formed by the extremities, but is inversely placed with respect to the line.

Laguerre also proved that an angle formed by any two lines real or imaginary is equal to $\frac{1}{2i} \log_e k$, in which k is the anharmonic ratio formed by the sides of the angle and the isotropic lines through its vertex.[1]

Another important theorem of Laguerre is:

The two isotropic lines passing through the focus of a conic

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[1]Oeuvres de Laguerre, Vol. II, p. 9.

are tangent to the conic and conversely.[1]

In 1873, Darboux[2] defined the Associate Points of two real points, A and B, as the two imaginary intersections, C and D, of the zero-circles about these real points. Also C and D are conjugate imaginary, lying on the perpendicular bisector of the line AB. Therefore we have the similarity with the method of representation of Laguerre that A and B represent the same points as compared with C and D as the representative points a and a' do with respect to A and A'.

Darboux made use of the Associate points in studying the properties of the generalized Lemniscates and Cassinians.

In 1907, E. W. Davis[3] of Nebraska University developed a method of representing an imaginary point $(x' + ix'', y' + iy'')$, with rectangular axes, by means of a rectilinear segment drawn from (x', y') to $(x' + x'', y' + y'')$ which he denoted by $\tilde{P}$.

The square of the distance between two imaginary points was developed from the usual formula and reduced to the same form as was developed by Laguerre. Davis extended the discussion of Imaginary straight lines in general and also the isotropic lines. He also made a study of the equations of conics in imaginary domains. In his study of the circle he discovered that certain curves are related to each other as follows:-

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1 Oeuvres de Laguerre Vol. II. p. 1.
2 "Sur une class remarquable de courbes et de surfaces algebriques". A. Hermann, Paris, 1873.
3 University Studies of University of Nebraska, Vol. X. No. 1. p. 1.

Taking as the fundamental curve, the double circle

$$x'^2 + y'^2 = a'^2 \text{ and } (x' + x'')^2 + (y' + y'')^2 = (a' + a'')^2,$$

the supplementary curve is $\tilde{x}_1^{\,2} - \tilde{y}_1^{\,2} = \tilde{a}^2$ and the companions to the supplementary curve are $x^2 - \frac{a'^2}{a''^2} y^2 = 1$, and

$x^2 - \left(\frac{a' + a''}{a' - a''}\right)y^2 = 1$. This completion of the circle $x^2 + y^2 = 1$ by the introduction of its complex points enables us to extend our notions of an angle and its functions.

Therefore we get $\quad \tilde{\theta} = \theta' + i\theta''$

$$\cos \tilde{\theta} = \cos \theta' \operatorname{ch} \theta'' - i \sin \theta'' \operatorname{sh} \theta'.$$

$$\sin \tilde{\theta} = \sin \theta' \operatorname{ch} \theta'' + i \cos \theta'' \operatorname{sh} \theta'.$$

$$\tan \tilde{\theta} = \frac{\tan \theta' + i \operatorname{th} \theta''}{1 - i \tan \theta' \operatorname{th} \theta''}.$$

In regard to the general conic $\frac{\tilde{x}^2}{a^2} + \frac{\tilde{y}^2}{b^2} = \tilde{k}^2$, the supplementary curves, when k^2 is real, are hyperbolas, whose pairs of asymptotes are pairs of conjugate diameters of the ellipse.

In April, 1911, G. B. Mathews presented a paper before the London Mathematical Society entitled, "A Cartesian Theory of Complex Geometrical Elements of Space."[1] In this discussion Mathews used the same representation as that used by Davis, and which he shows is somewhat related to that developed by Von Standt. In developing the equation for the square of the distance between two imaginary points, Mathews used the usual formula and obtained the following result:--

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[1]Proceedings of London Mathematical Society. Series 2. Vol. X. Part. 3. Pages 173 - 181.

$$\overline{AB\cdot CD}^2 = (\overline{AD}^2 + \overline{BC}^2 - \overline{AB}^2 - \overline{CD}^2 - \overline{BD}^2) + (\overline{AD}^2 + \overline{BC}^2 - \overline{AB}^2 - \overline{CD}^2 - 2\,\overline{AC}^2)\,i.$$

He introduced the idea of the "strip-line" which he defines as a complex line $[(AB), (CD)]$ such that every point on it is represented by a segment joining a real point on AC to a real point on BD and conversely.

In his consideration of the area of a triangle with vertices (A_1B_1) (A_2B_2) (A_3B_3), he found the area of the complex triangle

$$\left[A_1B_1 \cdot A_2B_2 \cdot A_3B_3\right] = \left[A_1A_2A_3\right] - \Sigma Q + i\left\{\left[B_1B_2B_3\right] - \left[A_1A_2A_3\right] - \Sigma Q\right\},$$

in which $[A_1A_2A_3]$ and $[B_1B_2B_3]$ are the areas of the real triangles and $\Sigma Q = \left[A_2A_3B_2B_3\right] + \left[A_3A_1B_3B_1\right] + \left[A_1A_2B_1B_2\right]$.

E. Study, in his "Vorlesungen über ausgewöhlte Gebiete der Geometrie" (Teubner, Leipzig, 1911), makes a systematic study of the imaginary elements in geometry and extends the ideas of Laguerre.